ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 13 Advanced Digital Communications
Homework 5 November 6, 2009

PROBLEM 1. (a) We know Fourier transform of sinc function is a rect. So the Fourier transform of sinc^2 is rect $*$ rect. The Fourier transform of

$$\text{sinc}^2(Wt)$$

is

$$\frac{1}{W}\Lambda(\frac{f}{W})$$

,where $\Lambda(f)$ is a triangle function zero valued at $|f| = 1$ and unit valued at $f = 0$.

(b)

$$v(t) = \sum_k u(KT)\text{sinc}(\frac{t}{T} - k) * \text{sinc}^2(Wt) = \frac{1}{2W}\sum_k u(KT)g(t - KT)$$

(c) Because both $u(KT)$ and $g(t - KT)$ are non negative then $v(t)$ is non negative.

(d) In formula $u(t) = \sum_k u(KT)\text{sinc}(\frac{t}{T} - k)$, we set $u(t) = 1$. The result follows.

(e) We take Fourier transform of

$$\sum_k g(\frac{t}{T} - k) = g(\frac{t}{T}) * \sum_k \delta(t - k)$$

We get

$$G(fT)\sum_k \delta(f - k) = 2\frac{1}{W}\delta(f)$$

so $\sum_k g(\frac{t}{T} - k) = 2$

(f)

$$v(t) = \sum_k u(kT)g(t - kT) \leq \sum_k g(t - kT) = 2$$

(g)

$$\begin{aligned} |v(t)| &= |\sum_k u(kT)g(t - kT)| && (1)\\ &\leq \sum_k |u(kT)||g(t - kT)| && (2)\\ &\leq \sum_k |g(t - kT)| && (3)\\ &= \sum_k g(t - kT) && (4)\\ &= 2 && (5) \end{aligned}$$

PROBLEM 2. (a) For $0 \leq t \leq \frac{1}{2}$, we have

$$p(t)p(t-1) = p(t) - p^2(t) = 0$$

So the inner product which is an integration is also zero.

(b) $p(t)p(t-k)$ has nonzero values only for $k = 1$. The rest follows from part (a).

(c) The case $|k| = 1$ and $m = 0$ easily follows from previous part. For the other values of k the product of two functions is zero everywhere.

(d) We know $p(t)$ is zero-one valued function which is symmetric with respect to $\frac{1}{2}$, so

$$\int_0^1 p^2(t)e^{j2\pi mt}\, dt = 0$$

(e) yes, this property will hold.

PROBLEM 3. (a) We want to minimized the MSE,

$$\text{MSE} = E(x^2) + E(\hat{x}^2) - 2E(x\hat{x})$$

so,

$$\text{MSE} = E(x^2) + a^2|c^t h|^2 E(x^2) + a^2\frac{N_0}{2} - 2a|c^t h|E(x^2)$$

By differentiating with respect to a and setting it equal to zero we will have the result.

(b) The value of MSE is

$$\frac{\frac{N_0}{2}E(x^2)}{|c^t h|^2 E(x^2) + \frac{N_0}{2}}$$

By simplifying above we will have the result.

(c) To minimize the MSE we should maximize the denominator, by Cauchy inequality we know

$$|c^t h|^2 \leq ||h||^2$$

and it satisfies the equality if h and c are proportional to each other, putting the norm constraint we will have $c = \frac{h}{|h|}$.

PROBLEM 4. We set $\hat{x} = A^t Y$. By orthogonality principle we have,

$$E((X - \hat{X})Y^*) = 0$$

So we have

$$E(XY^*) = E(A^t Y Y^*)$$

So in general we will have

$$\begin{bmatrix} E(Y_1Y_1^*) & E(Y_1Y_2^*) \\ E(Y_2Y_1^*) & E(Y_2Y_2^*) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} E(XY_1^*) \\ E(XY_2^*) \end{bmatrix}$$

Which is

$$\begin{bmatrix} \mathcal{E}_x + E(Z_1Z_1^*) & \mathcal{E}_x + E(Z_1Z_2^*) \\ \mathcal{E}_x + E(Z_2Z_1^*) & \mathcal{E}_x + E(Z_2Z_2^*) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \mathcal{E}_x \\ \mathcal{E}_x \end{bmatrix}$$

We solve the above linear equations for each case

(a)

$$a_1 = a_2 = \frac{\mathcal{E}_x}{2\mathcal{E}_x + 1}$$

(b)

$$a_1 = a_2 = \frac{\mathcal{E}_x}{2\mathcal{E}_x + 1 + \frac{\sqrt{2}}{2}}$$

(c) In this case the set of linear equations has infinite solutions, any of them is good for us. In particular

$$a_1 = a_2 = \frac{\mathcal{E}_x}{2(\mathcal{E}_x + 1)}$$