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School of Computer and Communication Sciences

Handout 11 Advanced Digital Communications
Homework 4 November 6, 2009

PROBLEM 1. (a)

$$\begin{aligned}
\Pr(R \le r) &= \Pr(|z| \le r) && (1)\\
&= \int\int P_z(x+iy)\,dx\,dy && (2)\\
&= \int_0^r \int_0^{2\pi} P_z(u\cos\theta + iu\sin\theta)u\,du\,d\theta && (3)\\
&= \int_0^r \int_0^{2\pi} P_z(ue^{i\theta})u\,du\,d\theta && (4)\\
& && (5)
\end{aligned}$$

Differentiating with respect to r will yield:

$$P_R(r) = r\int_0^{2\pi} P_z(re^{i\theta})\,d\theta$$

(b)

$$\Pr(U \le u) = \Pr(R^2 \le u) = \Pr(R \le \sqrt{u})$$

Differentiating with respect to u:

$$P_U(u) = \frac{1}{2\sqrt{u}}P_R(\sqrt{u}) = \frac{1}{2}\int_0^{2\pi} P_z(\sqrt{u}e^{i\theta})\,d\theta$$

(c) Since z is circularly symmetric, $P_z(\sqrt{u}e^{i\theta})$ does not depened on θ.

$$P_U(u) = \pi P_Z(\sqrt{u})$$

(d) If x and y are independent and with common density p, we have:

$$P_z(x+iy) = P_z(\sqrt{x^2+y^2}(\cos\phi + i\sin\phi)) = p(x)p(y)$$

Using part (c), we have

$$\begin{aligned}
P_U(x^2+y^2) &= \pi P_z(\sqrt{x^2+y^2}) && (6)\\
&= \pi p(x)p(y) && (7)
\end{aligned}$$

(e) Evaluating $P_U(x^2+y^2)$ at $x=0$ and $y=0$, we would have

$$P_U(x^2+y^2) = \frac{\pi P_u(x^2)P_u(x^2)}{(\pi p(0))^2}$$

Let us define $f(y) = \frac{P_U(u)}{(\pi p(0))^2}$. This is a continous function and satifies $f(a+b) = f(a)f(b)$ for all nonnegative a and b. Using hint we have $f(a) = e^{\beta a}$. Solving for β by integrating $P_U(u)$ and making it equal to 1.

$$\beta = -\pi p^2(0)$$

(f) Combining above we have

$$P_Z(z) = P_Z(|z|) = \frac{1}{\pi} P_U(|z|^2) = \frac{1}{\pi} e^{-\frac{|z|^2}{\sigma^2}}$$

So Z is a Gaussian random variable.

PROBLEM 2.

(a) By Markov bound, for any positive s, we have

$$\Pr(Z \geq b) = \Pr(e^{sZ} \geq e^{sb}) \leq E\big(e^{s(Z-b)}\big), \qquad s \geq 0.$$

(b)

$$\begin{aligned} Q(x) &= \Pr(z \geq x) && (8) \\ &= \frac{E\big(e^{sZ}\big)}{e^{sx}} && (9) \\ &= \frac{e^{\frac{s^2}{2}}}{e^{sx}} && (10) \\ &= e^{-\frac{x^2}{2}} && (11) \end{aligned}$$

In last step, we used the fact that in order to minimize the exponent we should take the value of $s = x$.

(c)

$$\begin{aligned} Q(x) &= \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{t^2}{2}}\, dt && (12) \\ &= \frac{1}{\sqrt{2\pi}} \int_x^\infty \frac{t}{t} e^{-\frac{t^2}{2}}\, dt && (13) \\ &= -\frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{t^2}{2}}}{t}\Big|_x^\infty - \frac{1}{\sqrt{2\pi}} \int_x^\infty \frac{1}{t^2} e^{-\frac{t^2}{2}}\, dt && (14) \\ &\leq \frac{1}{\sqrt{2\pi x^2}} e^{-\frac{x^2}{2}} && (15) \\ & && (16) \end{aligned}$$

For upperbound we have

$$\begin{aligned} Q(x) &= \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{t^2}{2}}\, dt && (17) \\ &= \frac{1}{\sqrt{2\pi}} \int_x^\infty \frac{t}{t} e^{-\frac{t^2}{2}}\, dt && (18) \\ &= -\frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{t^2}{2}}}{t}\Big|_x^\infty - \frac{1}{\sqrt{2\pi}} \int_x^\infty \frac{t}{t^3} e^{-\frac{t^2}{2}}\, dt && (19) \\ &= -\frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{t^2}{2}}}{t}\Big|_x^\infty + \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{t^2}{2}}}{t^3}\Big|_x^\infty + \frac{1}{\sqrt{2\pi}} \int_x^\infty \frac{1}{t^4} e^{-\frac{t^2}{2}}\, dt && (20) \\ &\geq \big(1 - \frac{1}{x^2}\big) \frac{1}{\sqrt{2\pi x^2}} e^{-\frac{x^2}{2}}. && (21) \end{aligned}$$

(d) Let $t = y + x$, and we have

$$Q(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \int_0^\infty e^{\frac{-y^2}{2} - xy}\, dy$$

It is obvious that

$$e^{\frac{-y^2}{2}} \leq 1$$

By mean value theorem and taylor expansion for some positive value y_*, we have

$$e^{\frac{-y^2}{2}} = 1 - \frac{y^2}{2} + \frac{y_*^4}{8} \geq 1 - \frac{y^2}{2}$$

We know

$$\int_0^\infty e^{-xy}\, dy = \frac{1}{x}$$

and

$$\int_0^\infty \frac{y^2}{2} e^{-xy}\, dy = \frac{1}{x^3}$$

Putting these facts together will give the bounds.

(e) We have

$$\Pr(|x_1| \leq x, |x_2| \leq x) = \Pr(|x_1| \leq x) \Pr(|x_2| \leq x) = (1 - 2Q(x))^2$$

(f) We have

$$\Pr(|x_1|^2 + |x_2|^2 \leq x) = \int_0^{2\pi} \int_0^x \frac{1}{2\pi} e^{-\frac{r^2}{2}}\, r\, dr\, d\theta = 1 - e^{-\frac{x^2}{2}}.$$

(g) Circle is contained in the square and we ignore $Q^2(x)$. We have the result.

PROBLEM 3. A baseband-equivalent waveform ($\omega_c > 2\pi$)

$$\tilde{x}_{bb}(t) = (x_1 + jx_2)\text{sinc}(t)$$

is convolved with the complex filter

$$w_1(t) = \delta(t) - j\delta(t-1)$$

(1)

$$y(t) = (x_1 + jx_2)\text{sinc}(t) - j(x_1 + jx_2)\text{sinc}(t-1)$$

(2)

$$\begin{aligned} z(t) &= w_2(t) * y(t) && (22) \\ &= 2j\text{sinc}(t) * ((x_1 + jx_2)\text{sinc}(t) - j(x_1 + jx_2)\text{sinc}(t-1)) && (23) \\ &= 2j(x_1 + jx_2)\text{sinc}(t) + 2(x_1 + jx_2)\text{sinc}(t-1) && (24) \end{aligned}$$

(3) We have

$$z_{bb}(t) = 2j(x_1 + jx_2)\text{sinc}(t) + 2(x_1 + jx_2)\text{sinc}(t-1)$$

and

$$\tilde{x}_{bb}(t) = (x_1 + jx_2)\text{sinc}(t)$$

and

$$\tilde{w}(t) = h_{bb}(t) = 4\text{sinc}(t-1) + j4\text{sinc}(t).$$

, which satify the equation

$$z_{bb}(t) = x_{bb}(t) * \frac{1}{2}h_{bb}(t)$$

PROBLEM 4. For mapping one we have

$$P_b = \frac{3}{2}Q(\frac{d}{2\sigma})$$

and for mapping two

$$P_b = Q(\frac{d}{2\sigma})$$

PROBLEM 5. (a)

$$f_{V|U}(\mathbf{v}|a) = \frac{1}{(\pi N_0)^n}e^{\frac{-||\mathbf{v}-a||^2}{N_0}}$$

and

$$f_{V|U}(\mathbf{v}|-a) = \frac{1}{(\pi N_0)^n}e^{\frac{-||\mathbf{v}+a||^2}{N_0}}$$

(b)

$$\text{LLR}(\mathbf{v}) = \log\frac{f_{V|U}(\mathbf{v}|-a)}{f_{V|U}(\mathbf{v}|a)} = \frac{-||\mathbf{v}-\mathbf{a}||^2 + ||\mathbf{v}+\mathbf{a}||^2}{N_0}.$$

(c) ML rule is comparing LLR to the constant zero and because LLR depends on difference of distance of channel output to vectors a and $-a$, ML is a minimum distance detector.

(d)

$$||\mathbf{v}-\mathbf{a}||^2 = ||\mathbf{v}||^2 - 2\langle\mathbf{v},\mathbf{a}\rangle + ||\mathbf{a}||^2$$

and

$$||\mathbf{v}+\mathbf{a}||^2 = ||\mathbf{v}||^2 + 2\langle\mathbf{v},\mathbf{a}\rangle + ||\mathbf{a}||^2$$

By substituting in (c) one gets the result.

(e) In the detection, only the real part of the $\langle\mathbf{v},\mathbf{a}\rangle$ matters and it is important if it is positive or negative, but $|\mathbf{v},\mathbf{a}|$ preserves none of the above.

(f) No, if v is in this space cv can be out of the space.