ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

**Handout 6** Advanced Digital Communications
Homework 2 October 9, 2009

PROBLEM 1.

(a)

$$R(H) = \int_Y \sum_j \sum_i \pi_j C_{i;j} \Pr(Y \in \Gamma_i | H_j) \quad (1)$$

$$= \sum_i \int_{Y \in \Gamma_i} \sum_j \pi_j C_{i;j} \Pr(Y \in \Gamma_i | H_j) \quad (2)$$

Now suppose, the decoder wants to assocate $Y = y$ to one of the decision regions $\Gamma_i$ so that the risk $R(H)$ is minimized. Therefore, the decoder chooses $y$ to be in $\Gamma_i$, in which $y$ minimizes $\sum_j \pi_j C_{i;j} \Pr(y|H_j)$

$$Y \in \Gamma_i : i = \underset{i}{\operatorname{argmin}} \sum_j \pi_j C_{i;j} \cdot \Pr(y|H_j)$$

(b) For the binary case the problem of finding the minimum is turned into a simple inequality checking. Therefore, we have

$$\sum_{j=0}^{1} \pi_j C_{0;j} \Pr(y|H_j) \underset{H_0}{\overset{H_1}{\gtrless}} \sum_{j=0}^{1} \pi_j C_{1;j} \Pr(y|Hj) \quad (3)$$

$$\pi_0 C_{0;0} \Pr(y|H_0) + \pi_1 C_{0;1} \Pr(y|H_1) \underset{H_0}{\overset{H_1}{\gtrless}} \pi_0 C_{1;0} \Pr(y|H_0) + \pi_1 C_{1;1} \Pr(y|H_1) \quad (4)$$

$$(\pi_1 C_{0;1} - \pi_1 C_{1;1}) \Pr(y|H_1) \underset{H_0}{\overset{H_1}{\gtrless}} (\pi_0 C_{1;0} - \pi_0 C_{0;0}) \Pr(y|H_0) \quad (5)$$

$$\frac{\Pr(y|H_1)}{\Pr(y|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0 (C_{1;0} - C_{0;0})}{\pi_1 (C_{0;1} - C_{1;1})} \quad (6)$$

The decision making only depends on the ratio $\Pr(y|H_1)/\Pr(y|H_0)$ and not the individual values of $\Pr(y|H_1)$ and $\Pr(y|H_0)$, and likelihood ratio is a sufficient statistics for optimal decision rule.

PROBLEM 2.

1. We know $C_{0;0} = C_{1;1} = 0$, $C_{1:1} = 1$ and $C_{0;1} = \frac{3}{4}$.

Following part (b) of Problem 1, we have:

$$\frac{\Pr(y|H_1)}{\Pr(y|H_0)} \overset{H_1}{\underset{H_0}{\gtrless}} \frac{\pi_0(C_{1;0} - C_{0;0})}{\pi_1(C_{0;1} - C_{1;1})} \quad (7)$$

$$\frac{e^{-2|y|}}{\frac{1}{2}e^{-|y|}} \overset{H_1}{\underset{H_0}{\gtrless}} \frac{\cancel{1/2}(1-0)}{\cancel{1/2}(\frac{3}{4}-0)} = \frac{4}{3} \quad (8)$$

$$e^{-|y|} \overset{H_1}{\underset{H_0}{\gtrless}} \frac{2}{3} \quad (9)$$

$$-|y| \overset{H_1}{\underset{H_0}{\gtrless}} \ln\frac{2}{3} \quad (10)$$

$$|y| \overset{H_0}{\underset{H_1}{\gtrless}} \ln\frac{3}{2} \quad (11)$$

We conclude

$$\Gamma_0 = \{ \quad y \in \mathbf{R} \quad | \quad |y| > \ln\frac{3}{2} \quad \} \quad (12)$$

$$\Gamma_1 = \{ \quad y \in \mathbf{R} \quad | \quad |y| < \ln\frac{3}{2} \quad \} \quad (13)$$

2. We use the same method we used in HW1, problem 6.

We define

$$\Gamma_{opt,0}(\pi_0) = \{y \in \mathbf{R} | (\pi_0 C_{1;0} - \pi_0 C_{0;0}) \Pr(y|H_0) \geq (\pi_1 C_{0;1} - \pi_1 C_{1;1}) \Pr(y|H_1)\} \quad (14)$$

$$\Gamma_{opt,1}(\pi_0) = \{y \in \mathbf{R} | (\pi_0 C_{1;0} - \pi_0 C_{0;0}) \Pr(y|H_0) < (\pi_1 C_{0;1} - \pi_1 C_{1;1}) \Pr(y|H_1)\} \quad (15)$$

and

$$\Gamma_{opt,0}(\pi_0') = \{y \in \mathbf{R} | (\pi_0' C_{1;0} - \pi_0' C_{0;0}) \Pr(y|H_0) \geq (\pi_1' C_{0;1} - \pi_1' C_{1;1}) \Pr(y|H_1)\} \quad (16)$$

$$\Gamma_{opt,1}(\pi_0') = \{y \in \mathbf{R} | (\pi_0' C_{1;0} - \pi_0' C_{0;0}) \Pr(y|H_0) < (\pi_1' C_{0;1} - \pi_1' C_{1;1}) \Pr(y|H_1)\} \quad (17)$$

We know that these regions are optimal decision regions for $R_{opt}(\pi_0)$ and $R_{opt}(\pi_0')$, respectively, which means they minimize $R$ for the given prior among all other decision regions. Now, we define

$$\pi_{c0} = \lambda\pi_0 + (1-\lambda)\pi_0' \quad (18)$$

$$\pi_{c1} = \lambda\pi_1 + (1-\lambda)\pi_1' \quad (19)$$

and

$$\Gamma_{opt,0}(\pi_c) = \{y \in \mathbf{R} | (\pi_{c0} C_{1;0} - \pi_{c0} C_{0;0}) \Pr(y|H_0) \geq (\pi_{c1} C_{0;1} - \pi_{c1} C_{1;1}) \Pr(y|H_1)\}$$

$$\Gamma_{opt,1}(\pi_c) = \{y \in \mathbf{R} | (\pi_{c0} C_{1;0} - \pi_{c0} C_{0;0}) \Pr(y|H_0) < (\pi_{c1} C_{0;1} - \pi_{c1} C_{1;1}) \Pr(y|H_1)\}.$$

and we conclude that

$$R_{opt}(\pi_0) = \int_Y \sum_j \sum_i \pi_j C_{i;j} \Pr(y \in \Gamma_i | H_j) \quad (20)$$

$$= \sum_{i=0}^{1} \int_{Y \in \Gamma_{opt,i}(\pi_0)} \sum_{j=0}^{1} \pi_j C_{i;j} \Pr(y \in \Gamma_{opt,i}(\pi_0) | H_j) \quad (21)$$

$$\leq \sum_{i=0}^{1} \int_{Y \in \Gamma_{opt,i}(\pi_c)} \sum_{j=0}^{1} \pi_j C_{i;j} \Pr(y \in \Gamma_{opt,i}(\pi_c) | H_j) \quad (22)$$

and

$$R_{opt}(\pi'_0) = \int_Y \sum_j \sum_i \pi'_j C_{i;j} \Pr(y \in \Gamma_i | H_j) \quad (23)$$

$$= \sum_{i=0}^{1} \int_{Y \in \Gamma_{opt,i}(\pi'_0)} \sum_{j=0}^{1} \pi'_j C_{i;j} \Pr(y \in \Gamma_{opt,i}(\pi'_0) | H_j) \quad (24)$$

$$\leq \sum_{i=0}^{1} \int_{Y \in \Gamma_{opt,i}(\pi_c)} \sum_{j=0}^{1} \pi'_j C_{i;j} \Pr(y \in \Gamma_{opt,i}(\pi_c) | H_j) \quad (25)$$

In (22) and (25), we used the fact that $\Gamma_{opt,0}(\pi_0), \Gamma_{opt,1}(\pi_0)$ and $\Gamma_{opt,0}(\pi'_0), \Gamma_{opt,1}(\pi'_0)$ minimize $R$ for the given prior among all other decision regions. Multiplying (22) by $\lambda$ and (25) by $1-\lambda$ and adding them up, we will have

$$\lambda R_{opt}(\pi_0) + (1-\lambda) R_{opt}(\pi'_0) \leq \sum_{i=0}^{1} \int_{Y \in \Gamma_{opt,i}(\pi_c)} \sum_{j=0}^{1} \lambda \pi_j C_{i;j} \Pr(y \in \Gamma_{opt,i}(\pi_c) | H_j) \quad (26)$$

$$+ \sum_{i=0}^{1} \int_{Y \in \Gamma_{opt,i}(\pi_c)} \sum_{j=0}^{1} (1-\lambda) \pi'_j C_{i;j} \Pr(y \in \Gamma_{opt,i}(\pi_c) | H_j)$$

$$= \sum_{i=0}^{1} \int_{Y \in \Gamma_{opt,i}(\pi_c)} \sum_{j=0}^{1} \pi_{cj} C_{i;j} \Pr(y \in \Gamma_{opt,i}(\pi_c) | H_j) \quad (27)$$

$$= R_{opt}(\pi_c) \quad (28)$$

Problem 3.

Following part (b) of Problem 1, we have:

$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-1)^2}{2\sigma^2}\right) \pi_1 N \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) \pi_0 \quad (29)$$

Clearly, if $N$ goes to infinity, for any given value of $\pi_1 \neq 0$, and $y$ and $\sigma^2$ finite, decoder chooses $H_1$.

We will have

$$\frac{(y-1)^2}{2\sigma^2} - \frac{y^2}{2} \underset{H_1}{\overset{H_0}{\gtrless}} \ln \frac{\pi_1 N}{\pi_0 \sigma} \quad (30)$$

$$(1-\sigma^2) y^2 - 2y + 1 \underset{H_1}{\overset{H_0}{\gtrless}} 2\sigma^2 \ln \frac{\pi_1 n}{\pi_0 \sigma} \quad (31)$$

Problem 4.

(a) Because of the additive nature of the channel, the error probability only depends on $P_N(\overrightarrow{y} - \overrightarrow{x_i})$. Now, if one shifts all $\overrightarrow{x_i}$'s by a constant vector, by shifting all decision regions by the same constant vector, one will design an equal error probability system for the second signal set.

(b)

$$\begin{aligned}
A' &= A - m(a) = \{a_j - m(a), \quad 1 \le j \le M\} && (32)\\
E(A') &= \frac{1}{M}\sum_j \langle (a_j - m(a)), (a_j - m(a))\rangle && (33)\\
&= \frac{1}{M}\sum_j \langle a_j, a_j\rangle + \frac{1}{M}\sum_j \langle m(a), m(a)\rangle - \frac{2}{M}\sum_j \langle a_j, m(a)\rangle && (34)\\
&= \frac{1}{M}\sum_j \langle a_j, a_j\rangle + \langle m(a), m(a)\rangle - 2\langle \frac{1}{M}\sum_j a_j, m(a)\rangle && (35)\\
&= \frac{1}{M}\sum_j \langle a_j, a_j\rangle - \langle m(a), m(a)\rangle && (36)\\
&= E(A) - \langle m(a), m(a)\rangle && (37)
\end{aligned}$$

By part (a), adding a constant vector $(-m(A))$ does not change the error probability, but it reduces the average transmitted energy, so it is good.

Problem 5.

(a) $V(R)$ for n-cube is $(2M)^n$, so number of signal points is $\frac{(2M)^n}{2^n} = M^n$.

$$\begin{aligned}
E(R) = \int_R ||x||^2 P(x)dx &= \sum_{k=1}^{n}\int_{-M}^{M} x_k^2 P(x_k)dx_k && (38)\\
&= n\frac{2M^2}{3}\frac{1}{2M} = n\frac{M^2}{3} && (39)
\end{aligned}$$

They are exact because a $n$-cube constellation of size $2M$ is the $n$-fold Cartesian product of an M-PAM constellation of the set of all odd integers in the interval [-M,M].

(b)

$$\text{Number of points : } \quad \frac{(\pi r^2)^{\frac{n}{2}}}{(\frac{n}{2})!2^n}$$

$$\text{Average energy : } \quad \frac{nr^2}{n+2}$$

(c) For $n = 2$ and same number of signal points, we have:

$$\begin{aligned}
M^2 &= \frac{\pi r^2}{4} && (40)\\
\frac{r^2}{M^2} &= \frac{4}{\pi} && (41)
\end{aligned}$$

So

$$\begin{aligned}
\frac{E_{\text{sphere}}}{E_{\text{cube}}} &= \frac{\frac{r^2}{2}}{\frac{2}{3}M^2} && (42)\\
&= \frac{r^2}{M^2}\cdot\frac{3}{4} = \frac{3}{\pi} = -0.2dB && (43)
\end{aligned}$$

(d)

$$M^{16} = \frac{(\pi r^2)^8}{8!2^16} \quad (44)$$

$$\Rightarrow M^2 = \frac{\pi r^2}{(8!)^{\frac{1}{8}}4} \quad (45)$$

$$\frac{E_{\text{sphere}}}{E_{\text{cube}}} = \frac{\frac{\cancel{16}r^2}{18}}{\frac{\cancel{16}M^2}{3}} \quad (46)$$

$$= \frac{r^2}{6M^2} = \frac{(8!)^{\frac{1}{8}} \cdot 4}{\pi \cdot 6} \approx -1dB \quad (47)$$

(e) We have

$$M^n = \frac{(\pi r^2)^{\frac{n}{2}}}{\frac{n}{2}!2^n} \quad (48)$$

$$\Rightarrow M^2 = \frac{\pi r^2}{4((\frac{n}{2}!))^{\frac{2}{n}}} \quad (49)$$

$$(50)$$

So

$$\frac{E_{\text{sphere}}}{E_{\text{cube}}} = \frac{\frac{n}{n+2}r^2}{n\frac{M^2}{3}} = \frac{3r^2}{(n+2)M^2} \quad (51)$$

$$= \frac{3}{n+2} \cdot \frac{4((\frac{n}{2})!)^{\frac{2}{n}}}{\pi} \quad (52)$$

$$= \frac{12}{(n+2)\pi} \cdot ((\frac{n}{2e})^{\frac{n}{2}})^{\frac{2}{n}} \cdot (\sqrt{2\pi\frac{n}{2}})^{\frac{2}{n}} \quad (53)$$

$$= \frac{6}{\pi e} \cdot \frac{n}{n+2}(\pi n)^{\frac{1}{n}} \quad (54)$$

$$\lim_{n\to\infty} \frac{E_{\text{sphere}}}{E_{\text{cube}}} = \frac{6}{\pi e} = -1.53dB$$