Handout 18 | Advanced Digital Communications
Homework 9 | November 30, 2009

PROBLEM 1. Consider the scalar discrete-time inter symbol interference channel considered in the class,

$$y_k = \sum_{n=0}^{\nu} p_n x_{k-n} + z_k, \quad k = 0, \dots, N-1, \tag{1}$$

where $z_k \sim \mathbf{C}\mathcal{N}(0, \sigma_z^2)$ and is i.i.d., independent of $\{x_k\}$. Let us employ a cyclic prefix as done in OFDM, *i.e.*,

$$x_{-l} = x_{N-1-l}, \quad l = 0, \dots, \nu.$$

As done in class given the cyclic prefix,

$$\mathbf{y} = \begin{bmatrix} y_{N-1} \\ \vdots \\ y_0 \end{bmatrix} = \underbrace{\begin{bmatrix} p_0 & \dots & \dots & p_\nu & 0 & \dots & 0 & 0 \\ 0 & p_0 & \dots & p_{\nu-1} & p_\nu & 0 & \dots & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \dots & \dots & 0 & p_0 & \dots & \dots & p_\nu \\ p_\nu & 0 & \dots & 0 & 0 & p_0 & \dots & p_{\nu-1} \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ p_1 & \dots & p_\nu & 0 & \dots & 0 & 0 & p_0 \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} x_{N-1} \\ \vdots \\ x_0 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} z_{N-1} \\ \vdots \\ z_0 \end{bmatrix}}_{\mathbf{z}}. \tag{2}$$

In the derivation of OFDM we used the property that

$$\mathbf{P} = \mathbf{F}^* \mathbf{D} \mathbf{F}, \tag{3}$$

where

$$\mathbf{F}_{p,q} = \frac{1}{\sqrt{N}} \exp\left(-j\frac{2\pi}{N}(p-1)(q-1)\right)$$

and $\mathbf{D}$ is the diagonal matrix with

$$\mathbf{D}_{l,l} = d_l = \sum_{n=0}^{\nu} p_n e^{-j\frac{2\pi}{N}nl}.$$

Using this we obtained

$$\mathbf{Y} = \mathbf{F}\mathbf{y} = \mathbf{D}\mathbf{X} + \mathbf{Z},$$

where $\mathbf{X} = \mathbf{F}\mathbf{x}$, $\mathbf{Z} = \mathbf{F}\mathbf{z}$. This yields the parallel channel result

$$\mathbf{Y}_l = d_l \mathbf{X}_l + \mathbf{Z}_l. \tag{4}$$

If the carrier synchronization is not accurate, then (1) gets modified as

$$y(k) = \sum_{n=0}^{\nu} e^{j2\pi f_0 k} p_n x_{k-n} + z_k, \quad k = 0, \dots, N-1 \tag{5}$$

where f_0 is the carrier frequency offset. If we still use the cyclic prefix for transmission, then (2) gets modified as

$$\underbrace{\begin{bmatrix} y(N-1) \\ \cdot \\ \cdot \\ y(0) \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} p_0 e^{j2\pi f_0 (N-1)} & \dots & p_\nu e^{j2\pi f_0 (N-1)} & 0 & \dots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & & \\ 0 & \ddots & \ddots & \ddots & e^{j2\pi f_0 \nu} p_0 & \dots & e^{j2\pi f_0 \nu} p_\nu \\ \ddots & \ddots & \ddots & \ddots & \ddots & & \\ e^{j2\pi f_0 0} p_1 & \dots & e^{j2\pi f_0 0} p_\nu & 0 & \dots & 0 & e^{j2\pi f_0 0} p_0 \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} x_{N-1} \\ \vdots \\ x_0 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} z_{N-1} \\ \vdots \\ z_0 \end{bmatrix}}_{\mathbf{z}}. \tag{6}$$

i.e.,

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{z}$$

Note that

$$\mathbf{H} = \mathbf{S}\mathbf{P},$$

where $\mathbf{S}$ is a diagonal matrix with $\mathbf{S}_{l,l} = e^{j2\pi f_0 (N-l)}$ and $\mathbf{P}$ is defined as in (2).

(a) Show that for $\mathbf{Y} = \mathbf{F}\mathbf{y}$, $\mathbf{X} = \mathbf{F}\mathbf{x}$,

$$\mathbf{Y} = \mathbf{G}\mathbf{X} + \mathbf{Z} \tag{7}$$

and prove that

$$\mathbf{G} = \mathbf{F}\mathbf{S}\mathbf{F}^*\mathbf{D}.$$

(b) If $f_0 \neq 0$, we see from part (a) that $\mathbf{G}$ is no longer a diagonal matrix and therefore we do not obtain the parallel channel result of (4). We get inter-carrier interference (ICI), *i.e.*, we have

$$\mathbf{Y}_l = \mathbf{G}_{l,l}\mathbf{X}_l + \underbrace{\sum_{q \neq l} \mathbf{G}(l,q)\mathbf{X}_q + \mathbf{Z}_l}_{\text{ICI + noise}}, \quad l = 0, \dots, N-1,$$

which shows that the other carriers interfere with $\mathbf{X}_l$. Compute the SINR (signal-to-interference plus noise ratio). Assume $\{\mathbf{X}_l\}$ are i.i.d, with $\mathbb{E}|\mathbf{X}_l|^2 = \mathcal{E}_x$. You can compute the SINR for the particular l and leave the expression in terms of $\{G(l,q)\}$.

(c) Find the filter $\mathbf{W}_l$, such that the MMSE criterion is fulfilled,

$$\min_{\mathbf{W}_l} \mathbb{E}|\mathbf{W}_l^* \mathbf{Y} - \mathbf{X}_l|^2.$$

You can again assume that $\{\mathbf{X}_l\}$ are i.i.d with $\mathbb{E}|\mathbf{X}_l|^2 = \mathcal{E}_x$ and that the receiver knows $\mathbf{G}$. You can now state the answer in terms of $\mathbf{G}$.

(d) Find an expression for $\mathbf{G}_{l,q}$ in terms of $f_0, N, \{d_l\}$. Given this and (b), what can you conclude about the value of f_0. For what values of f_0 do you think that the inter-carrier interference problem is important?

Hint: Use the summation of the geometric series

$$\sum_{n=0}^{N-1} r^n = \frac{1 - r^N}{1 - r}.$$

PROBLEM 2. Let we transmit data over an ISI channel, using OFDM strategy and cyclic prefix as you have seen in the class. So, we have

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{z},$$

where $\mathbf{x}$ and $\mathbf{y}$ are vectors of length $N+\nu$ and N, respectively. The source symbols are i.i.d. and from the constellation $\{+\sqrt{\mathcal{E}_x}, -\sqrt{\mathcal{E}_x}\}$, and the noise is additive white Gaussian with variance N_0 and independent of the source.

(a) What is the rate of transmission?

(b) In order to increase the rate, we approximate the channel with another ISI channel with a short length impulse response (called target channel) as shown in Figure(1). The goal is to find an optimum feedforward filter $\mathbf{W}$ such that the difference between the outputs of two channels is being minimized, *i.e.*,

$$\min_{\mathbf{W}} |r_k - u_k|^2$$

Find the expression of the optimum feedforward filter for a given target channel $\mathbf{B}$.

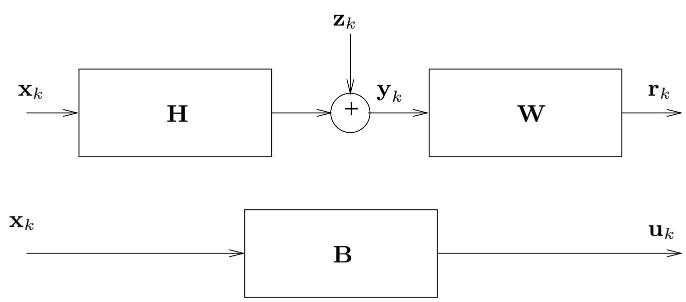

Figure 1: target response

(c) Find the SNR at the output of each of the systems and compare their performance.

PROBLEM 3. Consider a finite impulse response channel

$$\mathbf{y}_k = \sum_{n=0}^{\nu} \mathbf{p}_n x_{k-n} + \mathbf{z}_k$$

where $\mathbf{y}_k, \mathbf{z}_k \in \mathbb{C}^2$ and $\mathbf{p}_n \in \mathbb{C}^2$, i.e they are 2-dimensional vectors. This could arise, for example, through Nyquist sampling.

(a) Suppose one observes a block of N samples of $\{\mathbf{y}_k\}$, $\mathcal{Y}_k = \begin{bmatrix} \mathbf{y}_k \\ \vdots \\ \mathbf{y}_{k-N+1} \end{bmatrix}$. Write down the relationship between $\mathcal{Y}_k$ and $\mathcal{X}_k = \begin{bmatrix} x_k \\ \vdots \\ x_{k-N+1} \\ \vdots \\ x_{k-N+1-\nu} \end{bmatrix}$ in the form

$$\mathcal{Y}_k = \mathbf{P}\mathcal{X}_k + \mathcal{Z}_k$$

where $\mathcal{Y}_k$, $\mathcal{Z}_k \in \mathbb{C}^{2N}$, $\mathcal{X}_k \in \mathbb{C}^{N+\nu}$, $\mathbf{P} \in \mathbb{C}^{2N\times(N+\nu)}$ by specifying the form of $\mathbf{P}$.

(b) Suppose we use a cyclic prefix, i.e

$$x_{k-N-l} = x_{k-l}, \; l = 0, \cdots, \nu - 1$$

Develop the equivalent model:

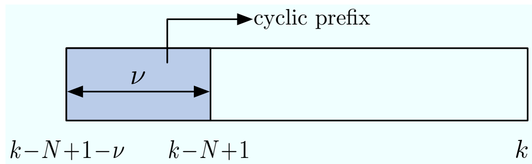

Figure 2: Cyclic Prefix

$$\mathcal{Y}_k = \widetilde{\mathbf{P}}\widetilde{\mathcal{X}}_k + \mathcal{Z}_k \tag{8}$$

where $\widetilde{\mathcal{X}}_k = \begin{bmatrix} x_k \\ \vdots \\ x_{k-N+1} \end{bmatrix} \in \mathbb{C}^N$ and $\widetilde{\mathbf{P}} \in \mathbb{C}^{2N \times N}$. Find $\widetilde{\mathbf{P}}$.

(c) Let $\mathbf{Y}(\ell) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathbf{y}_k e^{-j\frac{2\pi}{N}k\ell}$, $\mathbf{Z}(\ell) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathbf{z}_k e^{-j\frac{2\pi}{N}k\ell}$, $\mathbf{P}(\ell) = \sum_{n=0}^{\nu} \mathbf{p}_n e^{-j\frac{2\pi}{N}n\ell}$, $X(\ell) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_k e^{-j\frac{2\pi}{N}k\ell}$.

Develop the vector OFDM form for (8), i.e, show that

$$\mathbf{Y}(\ell) = \mathbf{P}(\ell)X(\ell) + \mathbf{Z}(\ell), \;\; \ell = 0, \cdots, N-1 \tag{9}$$

This can be done by either arguing about equivalent periodic sequences or any other proof technique. Here we would like to see a derivation, just stating the result is not enough.

(d) In the form given in (9), we get N parallel vector channels. If we want to detect each component $\{X(\ell)\}$ separately what would be the best linear estimator of $X(\ell)$ from $\mathbf{Y}(\ell)$, i.e the appropriate "frequency-domain" MMSE linear equalizer (FEQ).

(e) For the form in (9), if we concatenate with powerful codes, what would be the total achievable rate and the corresponding power optimization problem formulation. You need only formulate the power optimization problem and not neccessarily solve it.

Hint: The achievable rate for a single AWGN channel under power constraint P is given by

$$R = \log(1 + SNR) = \log(1 + \frac{P}{N_0})$$

PROBLEM 4. Given an OFDM system with $N = 4$ sub-carriers, cyclic prefix and a given channel $P(D) = 1 + 1.81D + 0.81D^2$:

1. Find the matrix $\mathbf{P}$ and compute its eigen decomposition.
2. After applying FEQ, compute the SNR on each sub carrier.

Assume that $E_x = 0.1$ and $\sigma^2 = 0.01$.