ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

**Handout 16** Advanced Digital Communications
Homework 8 November 23, 2009

PROBLEM 1. Consider the channel

$$y(k) = ||p||x(k) * q(k) + z(k),$$

where $q(k) = \delta(k) + b\delta(k-1) + b\delta(k+1)$ and $z(k)$ is zero-mean Gaussian noise with power spectral density (PSD) $S_z(D) = N_0 Q(D)$. Assume that

$$b = \sqrt{\frac{N_0}{\mathcal{E}_x ||p||^2}} = \frac{1}{2}.$$

Remember that by definition, $SNR_{MFB} = \frac{\mathcal{E}_x ||p||^2}{N_0}$.

(a) We first consider a zero-forcing equalizer (**ZFE**).

(i) Draw the block diagram of this equalizer.

(ii) Give $W_{ZFE}(D)$. Find the poles of this function. Is $W_{ZFE}$ a stable filter?

(iii) Can you compute the noise variance per real dimension $\bar{\sigma}^2_{ZFE}$? From your observation in (ii), you should be able to predict the behaviour of $\bar{\sigma}^2_{ZFE}$.

(iv) Give the value of $SNR_{ZFE}$. What does this value mean?

(b) Now we consider a minimum-mean-squared-error linear equalizer (**MMSE-LE**).

(i) What is the block diagram for this equalizer?

(ii) Give $W_{MMSE-LE}(D)$.
*Hint.*

$$W_{MMSE-LE}(D) = \frac{||p||\mathcal{E}_x}{||p||^2 Q(D)\mathcal{E}_x + N_0}$$

(iii) Write down $V(D)$ as defined below and compute $v(k)$.
*Hint.* You will have to decompose $V(D)$ into its causal and anticausal part. Make sure that the region of convergence (ROC) of $V(D)$ contains the unit circle. Define

$$R(D) = W_{MMSE-LE}(D)Y(D)$$

and define $V(D)$ such that

$$R(D) = (1 - V(D))X(D) + Z'(D),$$

writing in time domain will yield

$$r_k = x_k + v_0 x_k + \sum_{n \neq 0} v_n x_{k-n} + z'_k$$

redefining

$$r_k = (1 - v_0)x_k + e'_k$$

(iv) Compute the unbiased $SNR_{MMSE-LE,U}$. How does this compare to $SNR_{ZFE}$?
*Hint.* We define

$$SNR_{MMSE-LE,U} = \frac{1}{v_0} - 1$$

This is because, one can compute

$$S_{EE}(D) = \mathcal{E}_x V(D)$$

and hence,

$$E(|e_k|^2) = \mathcal{E}_x v_0$$

and

$$E(|e'_k|^2) = E(|e_k|^2) - \mathcal{E}_x v_0^2$$

(c) Now we consider minimum-mean-squared-error decision-feedback equalization (**MMSE-DFE**).

(i) Draw the block diagram for a MMSE-DFE equalizer.

(ii) Write down $Q(D) + \frac{1}{SNR_{MFB}}$ and show that this expression can be factorized as $\gamma_0 G(D)G^*(D^{-*})$. Find the causal monic function $G(D)$ as well as $\gamma_0$. Make sure $\frac{1}{G(D)}$ is stable (ROC must contain the unit circle).

(iii) What is the optimal feedbackward filter $B_{\text{opt}}(D)$?
*Hint.* $1 - B(D)$ is feedbackward filter, hence $B(D)$ is causal and monic, i.e.,

$$B(D) = 1 + b_1 D + b_2 D^2 + \dots,$$

and if

$$Q(D) + \frac{1}{\text{SNR}_{\text{MFB}}} = \gamma_0 G(D)G^*(D^{-*})$$

in which $G(D)$ is monic and causal and $G^*(D^{-*})$ is monic and anti-causal, then

$$B_{\text{opt}}(D) = G(D):$$

(iv) What is the optimal feedforward filter $W_{\text{opt}}(D)$?
*Hint.*

$$W_{\text{opt}}(D) = B_{\text{opt}}(D) W_{MMSE-LE}(D)$$

(v) Write down $V(D)$ as defined below similar to previous part (for MMSE-DFE).
*Hint.*

$$V(D) = \frac{\frac{1}{SNR_{MFB}}}{\gamma_0 G^*(D^{-*})}$$

(vi) Using hint below, compute $v(0)$.
*Hint.* $G^*(D^{-*})$ is monic and anti-causal, therefore so is $\frac{1}{G^*(D^{-*})}$, so

$$v_0 = \frac{1}{SNR_{MFB}} \frac{1}{\gamma_0}$$

(vii) Finally, give the unbiased signal-to-noise ratio $SNR_{MMSE-DFE,U}$. *Hint.* We define

$$SNR_{MMSE-DFE,U} = \frac{1}{v_0} - 1$$

(viii) Compare this signal-to-noise ratio to $SNR_{MMSE-LE,U}$.

PROBLEM 2.

(a) For the channel of problem 2, homework 7, show that the canonical factorization is

$$Q(D) + \frac{1}{\mathrm{SNR}_{\mathrm{MFB}}} = \gamma_0(1 - r_2 D^{-1})(1 - r_2^* D).$$

What is $\gamma_0$ in terms of $a$ and $b$? Please do not do this from scratch. You have done much of the work for this in problem 2, HW7.

(b) Find $B(D)$ and $W(D)$ for the MMSE DFE.

(c) Give an expression for $\gamma_{\text{MMSE-DFE}}$ MMSE-DFE. Compute its values for $a = 0, .5, 1$ for the $E_x = 1$ and $\sigma^2 = 0.1$ Sketch $\gamma_{\text{MMSE-DFE}}$ as in problem 2, HW7. Compare with your sketches from problem 2, HW7.

*Hint.*

$$\gamma_{\text{ZFE}} = 10 \log_{10} \frac{\mathrm{SNR}_{\mathrm{MFB}}}{\mathrm{SNR}_{\text{MMSE-DFE}}}$$