ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 15 Advanced Digital Communications
Homework 7 November 16, 2009

Hint. We considered both discrete Fourier transform and D-transform in the class. In terms of Fourier transform we derived that the power spectral densities of input z_k and output $\hat{z}_k$ of the filter g_k satisfy the following:

$$\hat{S}(\theta) = |G(\theta)|^2 S(\theta)$$

where

$$\begin{aligned} R_k &= E[Z_j Z_{j+k}^*] \\ S(\theta) &= \sum_k R_k e^{-j2\pi\theta k} \\ E[|z_k|^2] &= \int_{-\frac{1}{2}}^{\frac{1}{2}} S(\theta)\, d\theta \end{aligned}$$

These formulas translate to the following in terms of D-transform,

$$S_{\hat{Z}}(D) = G(D)G^*(D^{-*})S_Z(D)$$

where

$$\begin{aligned} R_k &= E[Z_j Z_{j+k}^*] \\ S_Z(D) &= \sum_k R_k D^k \\ E[|z_k|^2] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} S_Z(e^{-j\omega})\, d\omega \end{aligned}$$

Problem 1. Suppose we have a linear time invariant channel, i.e.,

$$Y(D) = ||p||Q(D)X(D) + Z(D);$$

with $Q(D) = Q^*(D^{-*})$. Also there is another process $U(D) = H(D)X(D)$; which we want to estimate.

(a) Given observations $\{y_k\}$, find the linear estimator

$$\hat{U}(D) = W(D)Y(D)$$

which minimizes the mean-squared error, i.e.,

$$W(D) = \text{argmin}_{W(D)} E||u_k - \hat{u}_k||^2$$

You can assume that $\{X_k\}$ and $\{Z_k\}$ are independent and that

$$S_x(D) = E_x$$

and

$$S_z(D) = N_0 Q(D).$$

(b) Given the optimum linear MMSE estimator given in part (a) we define the error as

$$e_k = u_k - \hat{u}_k$$

. Find the power spectral density of $\{e_k\}$, $S_E(D)$.

(c) If $H(D) = 1$, can you comment on the operation performed in part (a)?

PROBLEM 2. Suppose we have a linear time invariant channel, i.e.,

$$Y(D) = ||p||Q(D)X(D) + Z(D);$$

with

$$\begin{aligned} ||p||^2 &= 1 + aa^* \\ Q(D) &= \frac{a^*D^{-1} + ||p||^2 + aD}{||p||^2} \\ 0 \leq |a| &< 1. \end{aligned}$$

(a) Find the zero forcing and minimum mean square error linear equalizers $W_{\text{ZFE}}(D)$ and $W_{\text{MMSE-LE}}(D)$. Use the variable $b = ||p||^2(1 + \frac{1}{\text{SNR}_{\text{MFB}}})$ in your expression for $W_{\text{MMSE-LE}}(D)$.

Hint.

$$\text{SNR}_{\text{MFB}} = \frac{||p||^2 E_x}{N_0}$$

Can you explain why?

(b) By substituting $e^{-j\omega} = D$ and taking $\text{SNR}_{\text{MFB}} = 10||p||^2$, use Matlab to plot (lots of samples of) $W(e^{j\omega})$ for both ZFE and MMSE-LE for $a = .5$ and $a = .9$. Discuss the differences between the plots.

(c) Find the roots r_1, r_2 of the polynomial

$$aD^2 + bD + a^*.$$

Show that $b^2 - 4aa^*$ is always a real positive number (for $|a| \neq 1$).

Hint. Consider the case where $\frac{1}{\text{SNR}_{\text{MFB}}} = 0$. Let r_2 be the root for which $|r_2| < |r_1|$. Show that $r_1 r_2^* = 1$.

(d) Use the previous results to show that for the MMSE-LE

$$W(D) = \frac{||p||}{a} \frac{D}{(D - r_1)(D - r_2)} = \frac{||p||}{a(r_1 - r_2)} \left(\frac{r_1}{D - r_1} - \frac{r_2}{D - r_2} \right).$$

(e) Show that for the MMSE-LE, $w_0 = \frac{||p||}{\sqrt{b^2 - 4aa^*}}$. By taking $\frac{1}{\text{SNR}_{\text{MFB}}} = 0$, show that for the ZFE, $w_0 = \frac{||p||}{1 - 4aa^*}$.

(f) For $E_x = 1$ and $\sigma^2 = 0.1$ find expressions for σ^2_{ZFE}, $\sigma^2_{\text{MMSE-LE}}$, γ_{ZFE} and $\gamma_{\text{MMSE-LE}}$

Hint.

$$\gamma_{\text{ZFE}} = 10 \log_{10} \frac{\text{SNR}_{\text{MFB}}}{\text{SNR}_{\text{ZFE}}}$$

and

$$\gamma_{\text{MMSE-LE}} = 10 \log_{10} \frac{\text{SNR}_{\text{MFB}}}{\text{SNR}_{\text{MMSE-LE}}}.$$

(g) Find γ_{ZFE} and $\gamma_{\text{MMSE-LE}}$ in terms of the paramter a and calculate for $a = 0, 0.5, 1$. Sketch γ_{ZFE} and $\gamma_{\text{MMSE-LE}}$for $0 \leq a < 1$.