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Handout 8 Advanced Digital Communications
Homework 5 October 16, 2009

PROBLEM 1. (Amplitude-limited functions) Sometimes it is important to generate baseband waveforms with bounded amplitude. This problem explores pulse shapes that can accomplish this

(a) Find the Fourier transform of $g(t) = \text{sinc}^2(2Wt)$. Show that $g(t)$ is bandlimited to $f \leq W$ and sketch both $g(t)$ and $\hat{g}(f)$. [*Hint.* Recall that multiplication in the time domain corresponds to convolution in the frequency domain.]

(b) Let $u(t)$ be a continuous real $\mathcal{L}_2$ function baseband-limited to $f \leq W$ (i.e. a function such that $u(t) = \sum_k u(kT)\text{sinc}(\frac{t}{T} - k)$, where $T = \frac{1}{2W}$. Let $v(t) = u(t) * g(t)$. Express $v(t)$ in terms of the samples $\{u(kT); k \in \mathcal{Z}\}$ of $u(t)$ and the shifts $\{g(t - kT); k \in \mathcal{Z}\}$ of $g(t)$. [*Hint.* Use your sketches in part (a) to evaluate $g(t) * \text{sinc}(\frac{t}{T})$.]

(c) Show that if the T-spaced samples of $u(t)$ are nonnegative, then $v(t) \geq 0$ for all t.

(d) Explain why $\sum_k \text{sinc}(\frac{t}{T} - k) = 1$ for all t.

(e) Using (d), show that $\sum_k g(\frac{t}{T} - k) = c$ for all t and find the constant c. [*Hint.* Use the hint in (b) again.]

(f) Now assume that $u(t)$, as defined in part (b), also satisfies $u(kT) \leq 1$ for all $k \in \mathcal{Z}$. Show that $v(t) \leq 2$ for all t.

(g) Allow $u(t)$ to be complex now, with $|u(kT)| \leq 1$. Show that $v(t) \leq 2$ for all t.

PROBLEM 2. (Orthogonal sets) The function $\text{rect}(\frac{t}{T})$ has the very special property that it, plus its time and frequency shifts, by kT and $\frac{j}{T}$, respectively, form an orthogonal set. The function $\text{sinc}(\frac{t}{T})$ has this same property. We explore other functions that are generalizations of $\text{rect}(\frac{t}{T})$ and which, as you will show in parts (a)–(d), have this same interesting property. For simplicity, choose $T = 1$.

These functions take only the values 0 and 1 and are allowed to be nonzero only over $[-1, 1]$ rather than $[-\frac{1}{2}, \frac{1}{2}]$ as with $\text{rect}(\frac{t}{T})$. Explicitly, the functions considered here satisfy the following constraints:

$$p(t) = p^2(t) \quad \text{for all}\, t \quad (0/1\, \text{property}); \tag{1}$$
$$p(t) = 0 \quad \text{for}\, |t| > 1; \tag{2}$$
$$p(t) = p(-t) \quad \text{for all}\, t \quad (\text{symmetry}); \tag{3}$$
$$p(t) = 1 - p(t-1) \quad \text{for}\, 0 \leq t \leq 1/2. \tag{4}$$

Note: because of property (3), condition (4) also holds for $\frac{1}{2} < t \leq 1$.

Note also that $p(t)$ at the single points $t = \pm\frac{1}{2}$ does not affect any orthogonality properties, so you are free to ignore these points in your arguments.

(a) Show that $p(t)$ is orthogonal to $p(t - 1)$.

Hint. Evaluate $p(t)p(t - 1)$ for each $t \in [0, 1]$ other than $t = \frac{1}{2}$.

(b) Show that $p(t)$ is orthogonal to $p(t-k)$ for all integer $k \neq 0$

(c) Show that $p(t)$ is orthogonal to $p(t-k)e^{j2\pi mt}$ for integer $k \neq 0$ and $m \neq 0$.

(d) Show that $p(t)$ is orthogonal to $p(t)e^{j2\pi mt}$ for integer $m \neq 0$.

Hint. Evaluate $p(t)e^{j2\pi mt} + p(t-1)e^{j2\pi m(t-1)}$.

(e) Let $h(t) = \hat{p}(t)$ where $\hat{p}(f)$ is the Fourier transform of $p(t)$. If $p(t)$ satisfies properties (1) – (4), does it follow that $h(t)$ has the property that it is orthogonal to $h(t-k)e^{j2\pi mt}$ whenever either the integer k or m is nonzero?

Note: almost no calculation is required in this problem.

PROBLEM 3. Consider estimating the real zero-mean scalar x from:

$$\mathbf{y} = \mathbf{h}x + \mathbf{w}$$

where $\mathbf{w} \sim \mathsf{N}(0, \frac{N_0}{2}\mathbf{I})$ is uncorrelated with x and h is a fixed vector in $\mathcal{R}^n$.

(a) Consider the scaled linear estimate $\mathbf{c}^t\mathbf{y}$ (with the normalization $||\mathbf{c}|| = 1$):

$$\hat{x} = a\mathbf{c}^t\mathbf{y} = (a\mathbf{c}^t\mathbf{h})x + a\mathbf{c}^t\mathbf{z} \tag{5}$$

Show that the constant a that minimizes the mean square error $(x-\hat{x})^2$ is equal to

$$\frac{\mathbb{E}[x^2]|\mathbf{c}^t\mathbf{h}|}{\mathbb{E}[x^2]|\mathbf{c}^t\mathbf{h}|^2 + \frac{N_0}{2}} \tag{6}$$

(b) Calculate the minimal mean square error (denoted by MMSE) of the linear estimate in (5) (by using the value of a in (6)). Show that

$$\frac{\mathbb{E}[x^2]}{\text{MMSE}} = 1 + \text{SNR} = 1 + \frac{\mathbb{E}[x^2]|\mathbf{c}^t\mathbf{h}|^2}{\frac{N_0}{2}} \tag{7}$$

For every fixed linear estimator $\mathbf{c}$, this shows the relationship between the corresponding SNR and MMSE (of an appropriately scaled estimate).

(c) In particular, relation (7) holds when we optimize over all $\mathbf{c}$ leading to the best linear estimator. Find the value of vector $\mathbf{c}$ (with the normalization $||\mathbf{c}|| = 1$) by minimizing the MMSE derived in part (b). Compute optimal MMSE.

Hint. Use Cauchy–Schwarz inequality.

PROBLEM 4. (Linear Estimation) Consider the additive noise model given below,

$$Y_1 = X + Z_1 \tag{8}$$

$$Y_2 = X + Z_2. \tag{9}$$

Let $X, Y_1, Y_2, Z_1, Z_2 \in \mathcal{C}$, i.e. they are complex random variables. Moreover, assume X, Z_1 and Z_2 are zero mean and Z_1 and Z_2 are independent of X.

(a) Assume the following: $\mathbb{E}[|X|^2] = \mathcal{E}_x$, $\mathbb{E}[|Z_1|^2] = \mathbb{E}[|Z_2|^2] = 1$ and $\mathbb{E}[Z_1 Z_2^*] = 0$. Given Y_1, Y_2 find the best minimum mean squared error linear estimator $\hat{X}$, where the optimization criterion is $\mathbb{E}[|X-\hat{X}|^2]$.

(b) If $\mathbb{E}[Z_1 Z_2^*] = \frac{1}{\sqrt{2}}$, what is the best MMSE linear estimator of X ?

(c) If $\mathbb{E}[Z_1 Z_2^*] = 1$, what is the best MMSE linear estimator of X ?