ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 7 Advanced Digital Communications
Homework 4 October 12, 2009

PROBLEM 1. Suppose Z is a complex random variable with density p_Z.

(a) Let $R = |Z|$. Show that the density p_R of R is given by

$$p_R(r) = r \int_0^{2\pi} p_Z(r \exp(j\theta))\, d\theta.$$

Hint: Write $\Pr(R \leq r)$ as an integral over x and y, then use polar coordinates.

(b) Let $U = R^2$. Show that its density is given by

$$p_U(u) = \frac{1}{2} \int_0^{2\pi} p_Z(\sqrt{u} \exp(i\theta))\, d\theta.$$

(c) Suppose now that Z is circularly symmetric. Show that

$$p_U(u) = \pi p_Z(\sqrt{u}).$$

(d) Again suppose Z is circularly symmetric. Let X and Y be its real imaginary parts. We know that X and Y are identically distributed, call the common density p. Suppose that X and Y are independent. Show that

$$p_U(x^2 + y^2) = \pi p(x) p(y).$$

(e) Under the assumptions of (d), conclude that

$$p_U(x^2 + y^2) = \frac{1}{\pi p(0)^2} p_U(x^2) p_U(y^2).$$

Assuming that p_U is continuous show that it must be given by

$$p_U(u) = \alpha \exp(-\alpha u), \quad u \geq 0,$$

where $\alpha = \pi p(0)^2$. Hint: The only continuous functions f that satisfies $f(a + b) = f(a)f(b)$ are those for which $f(a) = \exp(\beta a)$ for some β.

(f) Show that if Z is circularly symmetric complex random variable with independent real and imaginary parts, then Z must be Gaussian.

PROBLEM 2.

(a) As discussed in HW1, Problem 3, the Markov bound on the probability that a real random variable Z exceeds b is given by

$$\Pr(Z \geq b) \leq \frac{E(Z)}{b}.$$

Use Markov bound to derive the Chernoff bound on the probability that a real random variable Z exceeds b is given by

$$\Pr(Z \geq b) \leq E\big(e^{s(Z-b)}\big), \qquad s \geq 0.$$

Hint. $e^{s(z-b)} \geq 1$ when $z \geq b$, and $e^{s(z-b)} \geq 0$ otherwise.

(b) Use the Chernoff bound to show that

$$Q(x) \leq e^{-\frac{x^2}{2}}$$

(c) Integrate by parts to derive the upper and lower bounds

$$Q(x) \leq \frac{1}{\sqrt{2\pi x^2}} e^{-\frac{x^2}{2}} \quad (1)$$

$$Q(x) \geq \left(1 - \frac{1}{x^2}\right) \frac{1}{\sqrt{2\pi x^2}} e^{-\frac{x^2}{2}}. \quad (2)$$

(d) Here is another way to establish these tight upper and lower bounds. By using a simple change of variables, show that

$$Q(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \int_0^{\infty} e^{\frac{-y^2}{2} - xy}\, dy$$

Then show that

$$1 - \frac{y^2}{2} \leq e^{\frac{-y^2}{2}} \leq 1$$

Putting these together, derive the bounds of part (c).

For (e)-(g), consider a circle of radius x inscribed in a square of side $2x$.

(e) Show that the probability that a two-dimensional iid real Gaussian random variable X with variance $\sigma^2 = 1$ per dimension falls inside the square is equal to $(1 - 2Q(x))^2$.

(f) Show that the probability that X falls inside the circle is $1 - e^{-\frac{x^2}{2}}$.

Hint. Write $p_X(x)$ in polar coordinates.

(g) Show that (e) and (f) imply that when x is large,

$$Q(x) \leq \frac{1}{4} e^{-\frac{x^2}{2}}.$$

PROBLEM 3. A baseband-equivalent waveform ($\omega_c > 2\pi$)

$$\tilde{x}_{bb}(t) = (x_1 + jx_2)\text{sinc}(t)$$

is convolved with the complex filter

$$w_1(t) = \delta(t) - j\delta(t-1)$$

(1) Find

$$y(t) = w_1(t) * \tilde{x}_{bb}(t)$$

(2) Suppose $y(t)$ is convolved with the imaginary filter

$$w_2(t) = 2j\text{sinc}(t)$$

to get

$$z(t) = w_2(t) * y(t) \quad (3)$$

$$= w_2(t) * w_1(t) * \tilde{x}_{bb}(t) \quad (4)$$

$$= w(t) * \tilde{x}_{bb}(t) \quad (5)$$

Find $z(t)$. Note that $\text{sinc}(t) * \text{sinc}(t-k) = \text{sinc}(t-k)$, k an integer.

(3) Let

$$\tilde{z}(t) = \text{Real}(z(t)e^{j\omega_c t}) = \tilde{w}(t) * x(t)$$

where $x(t) = \text{Real}(\tilde{x}_{bb}(t)e^{j\omega_c t})$. Show that

$$\tilde{w}(t) = 4\text{sinc}(t-1)\cos(\omega_c t) - 4\text{sinc}(t)\sin(\omega_c t)$$

when convolved with the passband $x(t)$ will produce $\tilde{z}(t)$. *Hint.* Use baseband calculations.

PROBLEM 4. Consider two 4-QAM systems with the same 4-QAM constellation

$$s_0 = 1+i, \quad s_1 = -1+i, \quad s_2 = -1-i, \quad s_3 = 1-i.$$

For each system, a pair of bits is mapped into a signal, but the two mappings are different:

$$\text{Mapping 1} \;:\; 00 \to s_0,\; 01 \to s_1,\; 10 \to s_2,\; 11 \to s_3 \tag{6}$$

$$\text{Mapping 2} \;:\; 00 \to s_0,\; 01 \to s_1,\; 11 \to s_2,\; 10 \to s_3 \tag{7}$$

The bits are independent and 0s and 1s are equiprobable, so the constellation points are equally likely in both systems. Suppose the signals are decoded by the minimum distance decoding rule, and the signal is then mapped back into the two binary digits. Find the error probability (in terms of the Q function) for each bit in each of the two systems.

PROBLEM 5. Let $\mathbf{Z} = (Z_1, \ldots, Z_n)^T$ be a vector of complex iid Gaussian rvs with iid real and imaginary parts, each $N(0, \frac{N_0}{2})$. The input $\mathbf{U}$ is binary antipodal, taking on values $\mathbf{a}$ or $-\mathbf{a}$, where $\mathbf{a} = (a_1, \ldots, a_n)^T$ is an arbitrary complex n-vector. The observation $\mathbf{V}$ is $\mathbf{U} + \mathbf{Z}$, and the probability density of $\mathbf{Z}$ is given by

$$f_Z(z) = \frac{1}{(\pi N_0)^n} e^{(\sum_{j=1}^{n} \frac{-|z_j|^2}{N_0})} = \frac{1}{(\pi N_0)^n} e^{\frac{-||Z||^2}{N_0}}.$$

(a) Give expressions for $f_{V|U}(\mathbf{v}|a)$ and $f_{V|U}(\mathbf{v}|-a)$.

(b) Show that the log likelihood ratio for the observation $\mathbf{v}$ is given by

$$\text{LLR}(\mathbf{v}) = \frac{-||\mathbf{v}-\mathbf{a}||^2 + ||\mathbf{v}+\mathbf{a}||^2}{N_0}.$$

(c) Explain why this implies that ML detection is minimum distance detection (defining the distance between two complex vectors as the norm of their difference).

(d) Show that $\text{LLR}(\mathbf{v})$ can also be written as $\frac{4\text{Re}(\langle \mathbf{v}, \mathbf{a} \rangle)}{N_0}$.

(e) The appearance of the real part, $\text{Re}(\langle \mathbf{v}, \mathbf{a} \rangle)$, in part (d) is surprising. Point out why log likelihood ratios must be real. Also explain why replacing $\text{Re}(\langle \mathbf{v}, \mathbf{a} \rangle)$ by $|\langle \mathbf{v}, \mathbf{a} \rangle|$ in the above expression would give a non-sensical result in the ML test.

(f) Does the set of points $\{\mathbf{v} : \text{LLR}(\mathbf{v}) = 0\}$ form a complex vector space?