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Handout 3 Advanced Digital Communications
Homework 2 September 25, 2009

PROBLEM 1. In class, we considered hypothesis testing with criterion of overall probibility of error,

$$P_{error} = \pi_0 \cdot P(\text{error}|H_0) + \pi_1 \cdot P(\text{error}|H_1).$$

Let us consider an application of hypothesis testing in radar, in this case H_1 represents the presence of a flying object and H_0 represents absence of a flying object, an error in detection of them is called false negative and false positive (false alarm) respectively. A system designer might decide that a false negative is much more dangerous than a false positive, so he modifies his criterion to the following formula,

$$R = \pi_0 C_{1,0} \cdot P(\text{error}|H_0) + \pi_1 C_{1,0} \cdot P(\text{error}|H_1),$$

where $C_{1,0} \gg C_{1,0} > 0$.

Following this discusion, error probability is just one possible criterion for choosing a detector. As you can see, more generally, the detectors minimize other cost functions. For example, let $C_{i;j}$ denote the cost of choosing hypothesis i when actually hypothesis j was true. Then the expected cost incurred by some decision rule $H(y)$ is:

$$R_j(H) = \sum_i C_{i;j} P[H(Y) = m_i | M = m_j]$$

Therefore the overall average cost after taking prior probabilities into account is:

$$R(H) = \sum_j \pi_j R_j(H)$$

(a) What is the optimal decision rule to minimize the above equation?

(b) Consider the binary case, and show that likelihood ratio $\frac{p(y|H_0)}{p(y|H_1)}$ is a sufficient statistics.

PROBLEM 2. Suppose Y is a random variable that under hypothesis H_j has density

$$P_j(y) = \frac{j+1}{2} e^{(j+1)|y|}, \qquad y \in \mathbf{R}, \qquad j = 0, 1.$$

Assume that costs are given by

$$C_{ij} = \begin{cases} 0 & \text{if} \quad i = j; \\ 1 & \text{if} \quad i = 1 \quad \text{and} \quad j = 0; \\ \frac{3}{4} & \text{if} \quad i = 0 \quad \text{and} \quad j = 1: \end{cases}$$

1. Find the optimum risk minimizing decision region assuming equal priors.

2. Recall that average risk function is given by:

$$R_H(\pi_0) = \sum_{j=0}^{1} \pi_j C_{0;j} + \sum_{j=0}^{1} \pi_j (C_{1;j} - C_{0;j}) P[H(Y) = m_1 | M = m_j]$$

Assume that costs are given as above. Show that $R_{opt}(\pi_0)$ is a concave function of π_0. Find the minimum, maximum value of $R_{opt}(\pi_0)$ and the corresponding priors.

Problem 3. Consider the simple hypothesis testing problem for the real-valued observation Y :

$$H_0 : p_0(y) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{y^2}{2}); y \in \mathbf{R} \quad (1)$$

$$H_1 : p_1(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(y-1)^2}{2\sigma^2}); y \in \mathbf{R} \quad (2)$$

Suppose the cost assignment is given by $C_{0;0} = C_{1;1} = 0$; $C_{1;0} = 1$, and $C_{0;1} = N$. Find the decision regions for optimal minimum risk detection and investigate the behaviour when N is very large.

Problem 4. Consider an arbitrary signal set $A = \{a_j, 1 \le j \le M\}$. Assume that all signals are equiprobable. Let $m(A) = \frac{1}{M}\sum_j a_j$ be the average signal, and let A' be A translated by $m(A)$ so that the mean of A' is zero:

$$A' = A - m(A) = \{a_j - m(A), 1 \le j \le M\}.$$

Let $E(A)$ and $E(A')$ denote the average energies of A and A', respectively.

(a) Show that the error probability of an optimum detector for an additive channel is the same for A' as it is for A.

(b) Show that $E(A') = E(A) - ||m(A)||^2$. Conclude that removing the mean $m(A)$ is always a good idea.

Problem 5. In this exercise we compare the power efficiency of n-cube and n-sphere signal sets for large n.

An n-cube signal set is the set of all odd-integer sequences of length n within an n-cube of side $2M$ centered on the origin. An n-sphere signal set is the set of all odd-integer sequences of length n within an n-sphere of squared radius r^2 centered on the origin.

Both n-cube and n-sphere signal sets therefore have minimum squared distance between signal points $d_{min}^2 = 4$ (if they are nontrivial), and n-cube decision regions of side 2 and thus volume 2^n associated with each signal point. The point of the following exercise is to compare their average energy using the following large-signal-set approximations:

- The number of signal points is approximately equal to the volume $V(R)$ of the bounding n-cube or n-sphere region R divided by 2^n, the volume of the decision region associated with each signal point (an n-cube of side 2).

- The average energy of the signal points under an equiprobable distribution is approximately equal to the average energy $E(R)$ of the bounding n-cube or n-sphere region R under a uniform continuous distribution.

(a) Show that if R is an n-cube of side $2M$ for some integer M, then under the two above approximations the approximate number of signal points is M^n and the approximate average energy is $\frac{nM^2}{3}$. Show that the first of these two approximations is exact.

(b) For n even, if R is an n-sphere of radius r, compute the approximate number of signal points and the approximate average energy of an n-sphere signal set, using the following expressions for the volume $V_\otimes(n, r)$ and the average energy $E_\otimes(n, r)$ of an n-sphere of radius r:

$$V_\otimes(n, r) = \frac{(\pi r^2)^{\frac{n}{2}}}{(\frac{n}{2})!} \quad (3)$$

$$E_\otimes(n, r) = \frac{nr^2}{n+2} \quad (4)$$

(c) For $n = 2$, show that a large 2-sphere signal set has about 0.2 dB smaller average energy than a 2-cube signal set with the same number of signal points.

(d) For $n = 16$, show that a large 16-sphere signal set has about 1 dB smaller average energy than a 16-cube signal set with the same number of signal points. [Hint: $8! = 40320$ (46.06 dB).]

(e) Show that as $n \to \infty$ a large n-sphere signal set has a factor of $\frac{\pi e}{6}$ (1.53 dB) smaller average energy than an n-cube signal set with the same number of signal points. [Hint: Use Stirling approximation, $m! \to (\frac{m}{e})^m \sqrt{2\pi m}$ as $m \to \infty$.]