ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 19 Advanced Digital Communications
Homework 10 December 14, 2009

PROBLEM 1. In a wireless setting with high scattering of the transmitted electromagnetic wave, the output of the channel can be modeled as follows.

$$y(t) = h(t)x(t),$$

where

$$h(t) = \sum_{l=1}^{L} h_l(t) = \sum_{l=1}^{L} C_l e^{j(2\pi \frac{v}{\lambda} \cos\theta_l t + \phi_l)},$$

where C_l is a random amplitude of the lth incoming wave, θ_l is its random angle with respect to the direction of the moving receive-antenna, and ϕ_l is a random phase-shift. We will investigate the statistical properties of $h(t)$. For this, we assume that C_l,θ_l and ϕ_l are mutually independent, and each i.i.d. for $l = 1, \ldots, L$. We further assume that C_l has second moment $\frac{\sigma^2}{L}$, that ϕ_l is uniformly distributed in $[0, 2\pi]$ and that θ_l is uniformly distributed in $[0, 2\pi]$. In a communication system, we will sample $y(t)$ at discrete time-steps. We will therefore analyze $h(t)$ at a fixed time instant t_0.

(a) For a given t_0, show that the expectation $\mathbb{E}[h_l(t_0)]$ is 0.
Hint: Use independence to write $h_l(t_0)$ as a product of expectations. It suffices to show that one of the factors is 0.

(b) Show that the variance of $h_l(t_0)$ is $\frac{\sigma^2}{L}$.

We will use the central limit theorem: If X_i, $i = 1, \ldots, L$ are i.i.d. random variables of any distribution with mean 0 and variance 1, then

$$\lim_{L\to\infty} \frac{1}{\sqrt{L}} \sum_{i=1}^{L} X_i \sim \mathcal{N}(0, 1),$$

i.e., the sum of L i.i.d. random variables converges to a random object that has Gaussian distribution.

(c) Define $X_l = \frac{\sqrt{N}}{\sigma} h_l(t_0)$ and apply the central limit theorem to X_l. Conclude that $h(t_0)$ behaves, for large L, as a Gaussian random variable of zero mean and variance σ^2.

After passing $y(t)$ through a matched filter, only the amplitude (the absolute value) of $h(t)$ will matter. The absolute value of a Gaussian random variable has a Rayleigh distribution. The factor $|h(t_0)|$ is called the Rayleigh fading factor.

PROBLEM 2. Assume that we are given the following values of a wireless communication system:

- mobile speed $v = 30\frac{m}{s}$
- carrier frequency $f_c = 3$GHz
- frequency bandwidth $W = 0.6$MHz

(a) Calculate the carrier wavelength λ.
Hint: The speed of light is $c = 3 \times 10^8 \frac{m}{s}$.

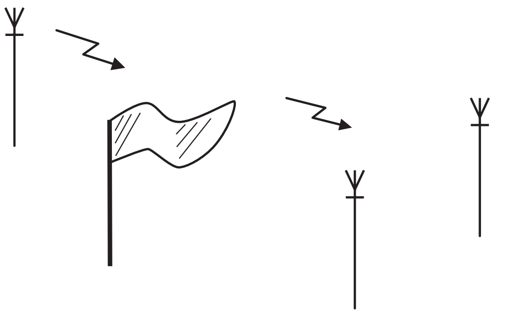
(b) Calculate the Doppler spread $f_{\max} = \frac{2v}{\lambda}$. What is the physical meaning of the Doppler spread?

(c) Calculate the coherence time T_c.

(d) We would like that the channel does not vary too much during the transmission of a block of N symbols. What is a reasonable range for the blocklength N?
Hint: The time required to transmit one symbol is roughly $T = \frac{1}{W}$.

(e) Assume now that the bandwidth is $W = 12$MHz. What is a reasonable range of N? Does this new range of N have an advantage compared to before?
Hint: Think about the number of symbols required to estimate the channel each time it changes.

PROBLEM 3. Suppose there is a transmitter which is sending signals to be received by two receive antennas. However due to a strange and unfortunate coincidence there is a flag fluttering in the wind quite close to one of the receive antennas and sometimes completely blocks the received signal.



In the absence of the flag, the received signal is given by a flat fading model, (discrete time model as done in class).

$$\mathbf{Y}_k = \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} h_1(k) \\ h_2(k) \end{bmatrix} x(k) + \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} \tag{1}$$

where $y_1(k)$, $y_2(k)$ are the received signals on first and second receive antennas respectively, $x(k)$ is the transmitted signal and $h_1(k)$, $h_2(k)$ are respectively the fading attenuation from the transmitter to the first and second receive antennas. Assume that $x(k)$ is binary, i.e. $x(k) \in \{-\sqrt{\mathcal{E}_x}, \sqrt{\mathcal{E}_x}\}$. The additive noise $z_1(k)$, $z_2(k)$ are assumed to be independent circularly symmetric complex Gaussian with variance (each) of σ^2. Assume that $h_1(k)$, $h_2(k)$ are i.i.d complex Gaussian $\mathbb{C}\eta(0,1)$.

1. Over several transmission blocks, compute the upper bound to the error probability and comment about the behavior of the error probability with respect to SNR for high SNR. *Hint*: Use the fact that $Q(x) \leq e^{-x^2/2}$.

2. Now let us consider the presence of fluttering flag which could potentially block only the second receive antenna. The model given in (1) now changes to:

$$\mathbf{Y}_k = \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} h_1(k) \\ \mathcal{F}_k h_2(k) \end{bmatrix} x(k) + \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix}$$

where:

$$\mathcal{F}_k = \begin{cases} 1 & \text{if there is no obstruction from the flag} \\ 0 & \text{if flag obstructs} \end{cases}$$

Suppose due to the random fluttering, the flag blocks a fraction q of the transmissions, i.e for a fraction q of the transmission, one receives only the signal from the first antenna.

Conditioned on $\mathcal{F}$, write down the error probabilities, i.e find and expression for $P_e(x \to x'|\mathcal{F})$ and compute its upper bound (see hint in (1)).

3. Find the overall error probability in the presence of the fluttering. How does the error probability behave at high SNR, i.e what diversity order does one obtain.

Hint: If the error probability behaves as $\frac{1}{SNR^D}$ at high SNR, the diversity order is D.