ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 2 Advanced Digital Communications
Homework 1 September 18, 2009

PROBLEM 1. One of the main issues in design of communication systems is channel modeling. The main goal of this question is to have you read about simple mathematical models and their validity for communication channels that we are going to consider throughout the course.

(a) There are many different impairments in a communication channel. Below is a short list of such models:

(1) Additive noise

(2) Distortion

(3) Interference

(4) Fading

(5) Doppler shift

In terms of resource, channels are generally characterized in mainly two types:

(i) Energy/Power limited

(ii) Bandwidth limited

Search the Internet or your own previous textbooks and write a brief one or two sentence description for above items.

Hint. You will find almost all answers in Wikipedia.

(b) Consider following channels:

(1) Wireline channels (non-optical)

(2) Optical channels

(3) Deep space satellite channels

(4) Microwave links

(5) Underwater acoustic channels

(6) Cellphone wireless communication channels

Categorize each channel based on characteristic of previous part.

Hint. Use the Hint from the previous part.

PROBLEM 2. Let V and W be discrete random variables(rvs) defined on some probability space with a joint pmf $p_{VW}(v, w)$.

(a) Prove that $E(V + W) = E(V) + E(W)$. Do not assume independence.

(b) Prove that if V and W are independent rvs, then $E(V \cdot W) = E(V) \cdot E(W)$.

(c) Assume that V and W are not independent. Find an example where $E(V \cdot W) \neq E(V) \cdot E(W)$ and another example where $E(V \cdot W) = E(V) \cdot E(W)$.

(d) Assume that V and W are independent and let σ_V^2 and σ_W^2 be the variances of V and W, respectively. Show that the variance of $V + W$ is given by $\sigma_{V+W}^2 = \sigma_V^2 + \sigma_W^2$.

PROBLEM 3.

(a) For a non-negative integer-valued rv N, show that

$$E(N) = \sum_{n>0} \Pr(N \geq n).$$

(b) Show, with whatever mathematical care you feel comfortable with, that for an arbitrary non-negative rv X,

$$E(X) = \int_0^\infty \Pr(X \geq a)\, da.$$

(c) Derive the Markov inequality, which says that for any $a > 0$ and any non-negative rv X,

$$\Pr(X \geq a) \leq \frac{E(X)}{a}.$$

Hint. Sketch $\Pr(X > a)$ as a function of a and compare the area of the rectangle with horizontal length a and vertical length $\Pr(X \geq a)$ in your sketch with the area corresponding to $E(X)$.

(d) Derive the Chebyshev inequality, which says that

$$\Pr\big(|Y - E(Y)| \geq b\big) \leq \frac{\sigma_Y^2}{b^2}$$

for any rv Y with finite mean $E(Y)$ and finite variance σ_Y^2.

Hint. Use part (c) with $(Y - E(Y))^2 = X$.

PROBLEM 4. Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent identically distributed (iid) analog rvs with the common probability density function $f_X(x)$. Note that $\Pr(X_n = \alpha) = 0$ for all α and that $\Pr(X_n = X_m) = 0$ for $m \neq n$.

(a) Find $\Pr(X_1 \leq X_2)$. (Give a numerical answer, not an expression; no computation is required and a one- or two-line explanation should be adequate.)

(b) Find $\Pr(X_1 \leq X_2; X_1 \leq X_3)$; in other words, find the probability that X_1 is the smallest of $\{X_1, X_2, X_3\}$. (Again, think — do not compute.)

(c) Let the rv N be the index of the first rv in the sequence to be less than X_1; i.e.,

$$\Pr(N = n) = \Pr(X_1 \leq X_2; X_1 \leq X_3; \ldots; X_1 \leq X_{n-1}; X_1 > X_n)$$

Find $\Pr(N \geq n)$ as a function of n.

Hint. Generalize part (b).

(d) Show that $E(N) = \infty$.

Hint. Use part (a) of Problem 3.

(e) Now assume that $X_1, X_2, \ldots$ is a sequence of iid rvs each drawn from a finite set of values. Explain why you can not find $\Pr(X_1 \leq X_2)$ without knowing the pmf. Explain why $E(N) = \infty$.

Problem 5. Let $X_1, X_2, \ldots, X_n$ be a sequence of n binary iid rvs. Assume that $\Pr(X_m = 0) = \Pr(X_m = 1) = \frac{1}{2}$. Let Z be a parity check on $X_1, X_2, \ldots, Xn$; i.e., $Z = X_1 \oplus X_2 \oplus \cdots \oplus X_n$ (where $0 \oplus 0 = 1 \oplus 1 = 0$ and $0 \oplus 1 = 1 \oplus 0 = 1$).

(a) Is Z independent of X_1? (Assume $n > 1$.)

(b) Are $Z, X_1, \ldots, X_{n-1}$ independent?

(c) Are $Z, X_1, \ldots, X_n$ independent?

(d) Is Z independent of X_1 if $\Pr(X_i = 1) \neq \frac{1}{2}$? (You may take $n = 2$ here.)

Problem 6. Consider the binary hypothesis testing problem with MAP decoding. Assume that priors are given by $(\pi_0, 1 - \pi_0)$.

(1) Let $V(\pi_0)$ be expectation of overall probability of error. Write the expression for $V(\pi_0)$.

(2) Show that $V(\pi_0)$ is a concave function of π_0 i.e.

$$V(\lambda\pi_0 + (1 - \lambda)\pi_0') \geq \lambda V(\pi_0) + (1 - \lambda)V(\pi_0')$$

for priors $(\pi_0, 1 - \pi_0)$ and $(\pi_0', 1 - \pi_0')$.

(3) What is the implication of concavity in terms of maximum of $V(\pi_0)$ for $\pi_0 \in [0, 1]$?

Problem 7. Consider the Gaussian hypothesis test case with non uniform priors. Prove that in this case, if y_1 and y_2 are elements of the decision region associated to hypothesis i then so is $\alpha y_1 + (1 - \alpha)y_2$, where $\alpha \in [0, 1]$.

Problem 8. Consider an additive-noise channel $Y = x + Z$, where x takes on the values ± 3 with $\Pr(x = 3) = \frac{1}{3}$ and where Z is a Cauchy random variable with PDF:

$$\Pr(Z = z) = \frac{1}{\pi(1 + z^2)}.$$

Determine the decision regions of the MAP detector. Compare the decision regions found with those of the MAP detector for $Z \sim N(0, 1)$. Compute the error probability in two cases (Cauchy and Gaussian noise).