EPFL Spring Semester 2009
Handout # 13, Mon, May 25, 2009

Homework Set # 5
Principles of Wireless Networks

Problem 1 (Time Expansion for Deterministic Non-layered Network)

Consider the deterministic network shown in Fig. 1. Note that this network is non-layered, *i.e.*, there are different paths from S to D with different length.

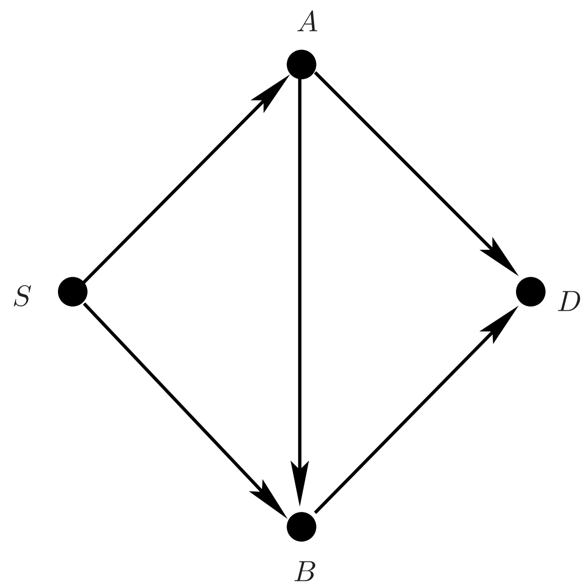

Figure 1: A non-layered deterministic network.

(a). Draw the expanded version of this network. Determine how the network behaves in the transition phases, *i.e.*, during the beginning and ending blocks.

(b). Analyze different cuts of the time-expanded network. Compute the cut values. Find the cuts corresponding to the cuts of the original network, and find the gap between the cut value in the original and time-expanded network.

Problem 2 (Compound Linear Deterministic Network)

Let $\mathcal{M}$ be the set of all binary $q \times q$ matrices of rank greater than or equal to r, *i.e*,

$$\mathcal{M} = \{\mathbf{M} \in \mathbb{F}_2^{q\times q} : \mathrm{rank}(\mathbf{M}) \geq r\}.$$

Also let $\mathcal{G}$ be a fixed linear deterministic network, with the set of nodes V and set of edges E. There is a source node S, and a destination node D in V. The network $\mathcal{N}$ is based on $\mathcal{G}$, *i.e.*, its topology is the same as that of $\mathcal{G}$. However, the channels realizations are chosen arbitrarily from the set $\mathcal{M}$ and kept fixed for the whole communication period. More precisely, the received message at node j at time instance k would be $\mathbf{y}_j[k] = \sum_{i\in V_j} \mathbf{M}_{ij}\mathbf{x}_i[k]$, where $V_j = \{i \in V : (i, j) \in E\}$, for some choice of $\{\mathbf{M}_{ij}\}$ made.

(a). Assume the transmitter does not know the channel realizations, but the receiver knows them. Write the upper bound to the capacity of the network, and evaluate it due to to the fact that the channel matrices are chosen from $\mathcal{M}$.

(b). Is the upper bound in (a) achievable? If yes, present an encoding and transmission scheme for that. If no, prove it.
Hint: Recall the compound channel result, that is, there exists a single codebook which is good enough for all the channel realizations.

Problem 3 (Ergodic Linear Deterministic Network)

In this problem we consider a time-varying network. Let $\mathcal{M}$ be the set of all binary $q \times q$ matrices.

Also let $\mathcal{G}$ be a fixed linear deterministic *layered* network, with the set of nodes V and set of edges E. There is a source node S, and a destination node D in V. The time-varying network $\mathcal{N}$ is based on $\mathcal{G}$ which means that its topology is the same as that of $\mathcal{G}$. However, the channel realizations is varying over blocks, that is they are chosen *uniformly at random* from the set $\mathcal{M}$ and kept fixed for the whole block of length T. At the end of the block they are again chosen independently from the previous block and uniformly at random from $\mathcal{M}$. More precisely, the received message at node j at time instance k in the block b, would be $\mathbf{y}_j^{(b)}[k] = \sum_{i \in V_j} \mathbf{M}_{ij}^{(b)} \mathbf{x}_i^{(b)}[k]$, where $V_j = \{i \in V : (i,j) \in E\}$.

(a). Assume the transmitter does not know the channel realizations, but the destination knows them. Write the upper bound to the ergodic capacity expression of the network.

(b). Is the upper bound in (a) achievable? If yes, present an encoding and transmission scheme for that. If not, prove it.
Hint: Note that the block length T can be assumed to be arbitrary large.

Problem 4 (Product Distribution vs. Arbitrary One)

Consider the deterministic network shown in Fig. 2, where the channels from S to both A and B are quaternary-input quaternary-output (two bits) determined by one-one functions

$$y_A[k] = f_{SA}\left(x_S[k]\right), \qquad y_B[k] = f_{SB}\left(x_S[k]\right).$$

The multiple access channel from A and B to D is binary-input ternary-output, with

$$y_D[k] = x_A[k] + x_B[k],$$

where here "$+$" denotes the real addition.

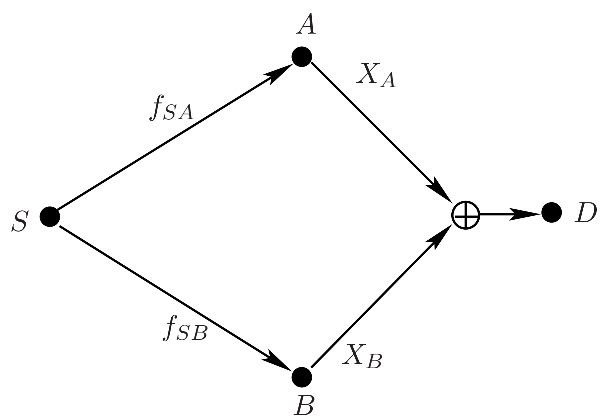

Figure 2: Real adder as a deterministic function.

(a). Evaluate all the cuts, and find cut-set upper-bound of the network.

(b). Analyze the transmission using independent mapping at the relays, *i.e.*, $p(X_S, X_A, X_B) = p(X_S)p(X_A)p(X_B)$. What the maximum rate can be achieved using this scheme?

(c). Evaluate $\max_{p(X_A,X_B)} I(X_A, X_B; Y_D)$. What is the maximizing distribution?

Consider the following encoding scheme. Denote the two bits send by the transmitter by (s, q). The transmitter generates 2^{RT} i.i.d. random pairs $(\mathbf{s}, \mathbf{q})$ according to $p_{S,Q}(s, q) = p_S(s)p_Q(q)$. It maps its message to one random pair and transmits it to the pair. Each relay has different codebooks for different received $\mathbf{q}$. The codebook used by A corresponding to $\mathbf{q}$ is generated randomly according to $p(\mathbf{x}_A|\mathbf{q}) = \prod_{i=1}^{T} p(x_A(i)|q(i))$ where

$$p_{X_A|Q}(x|q=0) = \begin{cases} 2/3 & x=0 \\ 1/3 & x=1 \end{cases} \qquad p_{X_A|Q}(x|q=1) = \begin{cases} 1/6 & x=0 \\ 5/6 & x=1 \end{cases}$$

Similarly, the conditional codebook of B is produced according to

$$p_{X_B|Q}(x|q=0) = \begin{cases} 3/4 & x=0 \\ 1/4 & x=1 \end{cases} \qquad p_{X_B|Q}(x|q=1) = \begin{cases} 0 & x=0 \\ 1 & x=1. \end{cases}$$

(d). Find the joint distribution p_{X_A,X_B} in terms of $\lambda = p_Q(q=0)$. For which value of λ, this distribution coincides with the maximizing one found in (c)?

(e). What is the maximum achievable rate of this scheme? Compare it to the upper bound of part (a). What is your conclusion?

Problem 5 (Partial Decode and Forward Strategy)

Consider the linear deterministic relay network shown in Fig. 3. Note that the network is not necessarily linear, however, the channels inputs and outputs are related through deterministic functions as

$$\mathbf{Y}_1 = f(\mathbf{X}) \qquad \mathbf{Y} = g(\mathbf{X}, \mathbf{X}_1).$$

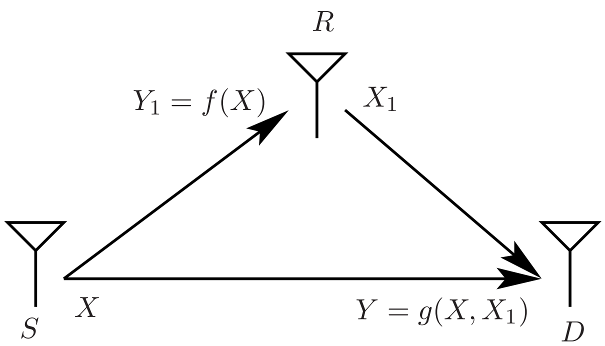

Figure 3: Deterministic relay network.

(a). Write the min-cut expression for this network, and find an upper bound for the admissible rate of the network.

In the following we will study a transmission scheme for this network, in which the relay decodes only a part of the transmitted message, and forwards it.

The transmitting message in block i is split into $m_i = (v_i, w_i)$, where $v_i \in \mathcal{V}$, $w_i \in \mathcal{W}$, and

$$\mathcal{V} = \{1, 2, \dots, 2^{nR_1}\}$$
$$\mathcal{W} = \{1, 2, \dots, 2^{nR_2}\}.$$

A binning process partitions the set $\mathcal{V}$ into 2^{nR_0} bins, each of size $2^{n(R_1-R_0)}$. We denote the bin indices by s and $c \in \mathcal{S} = \{1, 2, \dots, 2^{nR_0}\}$.

For each $s \in \mathcal{S}$, generate a length n random sequence $\mathbf{x}_1(s)$, according to the distribution $p_{X_1}(\mathbf{x}_1) = \prod_{t=1}^{n} p_{X_1}(x_1(t))$. Now, for each $s \in \mathcal{S}$ and each $v \in \mathcal{V}$ generate n i.i.d. sequences $\mathbf{u}(v|s)$ according to $p_{U|X_1}(\mathbf{u}) = \prod_{t=1}^{n} p_{U|X_1}(u(t)|x_1(t))$. Note that we will have 2^{nR_0} codebooks, each of size 2^{nR_1}. Similarly, we create different codebooks $w \in \mathcal{W}$. For each $s \in \mathcal{S}$, $v \in \mathcal{V}$ and $w \in \mathcal{W}$, generate length n sequences $\mathbf{x}(w|s,v)$ according to $p_{X|U,X_1}(\mathbf{x}) = \prod_{t=1}^{n} p_{X|U,X_1}(x(t)|u(t), x_1(t))$.

The transmission is done over B blocks. In block i, the transmitter finds the set index of v_{i-1}, which is s_{i-1}. Then in order to transmit (v_i, w_i), it looks for the codeword corresponding to w_i in the codebook assigned to v_i and s_{i-1}, that is $\mathbf{x}(w_i|v_i, s_{i-1})$, and transmits it.

At the end of block i, the relay decodes v_i. Then it looks for the index of the bin v_i belongs to, which is in fact s_i. Then it sends the corresponding codeword $\mathbf{x}(s_i)$ to the destination in block $i+1$.

At the end of block i, the destination has a sequence $\mathbf{y}(i)$, and creates a list, say $\mathscr{L}(\mathbf{y}(i))$ of all v's who are jointly typical with $\mathbf{y}(i)$. It keeps the list until end of block $i+1$, where it can decode s_i, from the message it has received from the relay. Then it looks in the list to find the $v \in \mathscr{L}(\mathbf{y}(i))$ whose bin index is s_i, and declares it as the v_i. Then having s_i and v_i, it focus on the corresponding codebook for w to decode it.

(b). Having $\mathbf{y}_1(i)$, what is the probability that a randomly chosen $\mathbf{u}$ be jointly typical with $\mathbf{y}(i)$? What is the maximum R_1, for which the relay can still decode v_i with high probability as n grows?

(c). As stated before, the decoder has to find the bin index s_i from what it heard from the relay. What is the maximum number of bins, such that this can be done with arbitrary small probability? Prove your answer using a typicality argument.

(d). Recall that the destination decodes v_i by intersecting $\mathscr{L}(\mathbf{y}(i))$ and the bin indexed by s_i. If such intersection has more than one member, this cannot be done with vanishing error probability. What is the requirement, in terms of the rates, which guarantees a unique point in this intersection?

(e). In the last decoding step of the block, the decoders has to find w_i assuming that it has already found s_i and v_i. What is the maximum R_2 for which w_i can be decoded with vanishing probability of error?

(f). Note that total rate of communication is $R = R_1 + R_2$. The transmitter can communicate $B-1$ pairs of (v, w) during the B blocks, which gives the transmission rate $(B-1)R/B$, which can be arbitrary close to R as B increases. Using the bounds derived in (b)-(e), what is the maximum rate of the proposed transmission strategy?

(g). Compare the achievable rate found in (f) to the upper-bound of part (a). Comment on its result.