Solutions: Homework Set # 3
Principles of Wireless Networks

Problem 1 (Multi Source Network)

(a). Let there exists an achievable rate tuple $\mathbf{R} = (R_1, \ldots, R_{|\mathcal{S}|})$ such that for a subset $\mathcal{T} \subseteq \mathcal{S}$, we have

$$\sum_{T\in\mathcal{T}} R_T > \min_{\Omega\in\Lambda(\mathcal{T},D)} C(\Omega).$$

We can create a new network by adding a "super-source" node to the original one A, that is $\mathcal{G}' = (V', E')$, where $V' = V \cup \{A\}$, and $E' = E \cup \{(A, j) : j \in \mathcal{T}\}$, with

$$c'_{ij} = \begin{cases} c_{ij} & \text{if } i \neq A, \\ N & \text{otherwise,} \end{cases}$$

where N is an integer larger than all the cut-values of the network. It is clear that A can communicate to D with rate at least $\sum_{T\in\mathcal{T}} R_T$. This can be done, by splitting its messages to sun-messages of rates $(R_T : T \in \mathcal{T})$, and send them to corresponding nodes in $\mathcal{T}$. The rest of the transmission follows exactly the transmission scheme used to achieve $\mathbf{R}$ in the original network. However, the max-flow min-cut theorem for one source implies that the maximum achievable rate can be send from A to D is

$$R_A = \min_{\Omega\in\Lambda(A,D)} C(\Omega).$$

It is clear that a cut with $\mathcal{T} \cap \Omega^c \neq \emptyset$, cannot be the minimizer since its value is at least N. The remaining cuts are the elements of $\Lambda(\mathcal{T}, D)$. Therefore, we have

$$\min_{\Omega\in\Lambda(\mathcal{T},D)} C(\Omega) < \sum_{T\in\mathcal{T}} R_T \leq \max R_A = \min_{\Omega\in\Lambda(A,D)} C(\Omega) = \min_{\Omega\in\Lambda(\mathcal{T},D)} C(\Omega),$$

which is a contradiction.

(b). In order to show the achievability, you just need to repeat the random argument used for single source. The only difference is that you have to consider different types of error for each cut, *i.e.,* the message of each source may be distinguishable or indistinguishable at each cut. Also note that if a node does not receive information from a source node T, then it can distinguish between different messages sent by T. Look at the solution of the second problem in Homework 4, for more details.

Problem 2 (Deterministic Relay Network in Half-Duplex Regime)

(a). There are two cuts, as follows:

- $\Omega_1 = \{S\}$: For this cut we have

$$\begin{bmatrix} \mathbf{y}_R \\ \mathbf{y}_D \end{bmatrix} = \mathbf{G}_{\Omega_1,\Omega_1^c} \mathbf{x}_S$$

where

$$\mathbf{G}_{\Omega_1,\Omega_1^c} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right].$$

Therefore $C(\Omega_1) = \text{rank}(\mathbf{G}_{\Omega_1,\Omega_1^c}) = 3$.

- $\Omega_2 = \{S,R\}$: Similarly,we can write

$$\mathbf{y}_D = \mathbf{G}_{\Omega_2,\Omega_2^c} \begin{bmatrix} \mathbf{x}_S \\ \mathbf{x}_R \end{bmatrix}$$

where

$$\mathbf{G}_{\Omega_2,\Omega_2^c} = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right].$$

Clearly $C(\Omega_2) = \text{rank}(\mathbf{G}_{\Omega_2,\Omega_2^c}) = 3$.

Hence the min-cut value is also 3.

(b). As we have seen in the class the min-cut value is achievable using random linear mapping at the relays, *i.e.,* $R = 3$ is achievable.

(c). Consider a block of of bits of length 5, $\mathbf{x} = [x_1, x_2, x_3, x_4, x_5]$. Consider tow consecutive transmission slots with

$$\mathbf{x}_S[t] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \mathbf{x}_S[t+1] = \begin{bmatrix} x_4 \\ x_5 \\ 0 \end{bmatrix}.$$

Let the relay listen to the source at time t and transmit to the receiver at $t+1$. Having received $\mathbf{x}_S[t]$, it transmits

$$\mathbf{x}_R[t+1] = \begin{bmatrix} x_3 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore the destination nodes receives

$$\mathbf{y}_D[t] = \begin{bmatrix} 0 \\ x_1 \\ x_2 \end{bmatrix}, \qquad \mathbf{y}_D[t+1] = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}.$$

Hence it can decode 5 bits using 2 channel uses, which gives us $R = 5/2$ bits/second.

(d). We assume $\alpha \in \mathbb{Q}$, otherwise one can find a rational number arbitrary close to it. Let $N \in \mathbb{N}$ be such that $M = N\alpha$ be an integer. In M transmission that the relay listens to the source, the destination and relay receive Mn_{SD} and Mn_{SR} bits, respectively. In the remaining $(N-M)$ seconds, the relay has only $M(n_{SR} - n_{SD})^+$ innovative bits to transmit to the destination, where $x^+ = \max(x,0)$. However, it can only send $(N-M)n_{RD}$ bits. Therefore, the destination nodes receives $A = \min[M(n_{SR} - n_{SD})^+, (N-M)n_{RD}]$ new bits from the relay. The source node can also transmits new bits to the destination if the destination can still decode something without interfering with the bits it receives from relay. If $n_{RD} \leq n_{SD}$, then receiving A bits from the relay, the destination can only receive $B = (N-M)n_{SD} - A$ bits from the source during the second phase of communication. If $n_{RD} \geq n_{SD}$, the destination can decode $(N-M)n_{RD} - A$ more bits, while the source can only send $(N-M)n_{SD}$ bits, *i.e.,* source can send $\min[(N-M)n_{RD} - A, (N-M)n_{SD}]$ new bits to the destination. It is easy to see both cases can be summarized in $\min[(N-M)\max(n_{RD}, n_{SD}) - A, (N-M)n_{SD}]$. So, the total number of bits received by the destination during N time slot is

$$\begin{aligned} K &= Mn_{SD} + A + \min[(N-M)\max(n_{RD}, n_{SD}) - A, (N-M)n_{SD}] \\ &= \min\left\{(N-M)\max(n_{RD}, n_{SD}) + Mn_{SD}, Nn_{SD} + M(n_{SR} - n_{SD})^+\right\}. \end{aligned}$$

Dividing by N we get the rate as

$$\begin{aligned} R(\alpha) &= \min\left\{(1-\alpha)\max(n_{RD}, n_{SD}) + \alpha n_{SD}, n_{SD} + \alpha(n_{SR} - n_{SD})^+\right\} \\ &= \min\left\{\max(n_{RD}, n_{SD}) - \alpha(n_{RD} - n_{SD})^+, n_{SD} + \alpha(n_{SR} - n_{SD})^+\right\}. \end{aligned}$$

(e). Define $R^* = \max_{\alpha \in [0,1]} R(\alpha)$. Note that it takes the minimum of two linear functions of α, where one is increasing and the other one is decreasing with respect to α. So, R^* is achieved when two functions are equal. After simplification we have

$$R^* = \begin{cases} \frac{n_{SR}n_{RD} - n_{SD}^2}{n_{SR} + n_{RD} - 2n_{SD}} & \text{if } n_{SD} < \min(n_{SR}, n_{RD}) \\ n_{SD} & \text{if } n_{SD} \geq \min(n_{SR}, n_{RD}). \end{cases}$$

Note that in the full-duplex regime we have $R = \min\{\max(n_{RD}, n_{SD}), \max(n_{SD}, n_{SR})\}$ which can be rewritten as

$$R = \begin{cases} \min(n_{SR}, n_{RD}) & \text{if } n_{SD} < \min(n_{SR}, n_{RD}) \\ n_{SD} & \text{if } n_{SD} \geq \min(n_{SR}, n_{RD}). \end{cases}$$

They are different iff $n_{SD} < \min(n_{SR}, n_{RD})$.

Problem 3 (Min-Cut value)

(a). Note that we can only focus on the connected cuts, that are cuts for which the network induced by Ω and Ω^c are connected. It is also worth mentioning that the matrix corresponding to each cut is a block diagonal one, where the number of blocks equals to the number of layers separated by the cut. Moreover, the rank of block diagonal matrices equal to the sum of the ranks of the block. In the following we list the value of $C(\Omega)$.

$$\begin{aligned} C(\{S, A_1\}) &= 3 + 4 = 7 \\ C(\{S, A_2\}) &= 5 + 3 = 8 \\ C(\{S, A_1, B_1\}) &= 3 + 4 + 5 = 12 \\ C(\{S, A_2, B_2\}) &= 5 + 2 + 2 = 9 \\ C(\{S, A_1, A_2, B_1\}) &= 4 + 5 = 9 \\ C(\{S, A_1, A_2, B_2\}) &= 2 + 2 = 4. \end{aligned}$$

For the layered cuts we have

$$C(\{S\}) = 5, \qquad C(\{S, A_1, A_2\}) = 6, \qquad C(\{S, A_1, A_2, B_1, B_2\}) = 5.$$

Therefore $\min_\Omega C(\Omega) = 4$.

(b). Note that the minimizing cut is $\Omega = \{S, A_1, A_2, B_2\}$, corresponding to

$$\mathbf{G}_{\Omega,\Omega^c} = \left[\begin{array}{c|c} \mathbf{G}_{A_1 B_1} + \mathbf{G}_{A_2 B_1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{G}_{B_2 D} \end{array} \right].$$

We have $\mathrm{rank}(\mathbf{G}_{\Omega,\Omega^c}) = \mathrm{rank}(\mathbf{G}_{A_1 B_1} + \mathbf{G}_{A_2 + B_1}) + \mathrm{rank}(\mathbf{G}_{B_2 D}$. One should choose $\mathbf{G}'_{A_2 B_1}$ such that the resulting matrix $\mathbf{G}_{A_1 B_1} + \mathbf{G}'_{A_2 + B_1})$ be of rank 3. This can be done by choosing (for example)

$$\mathbf{G}'_{A_2 B_1} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$