Homework Set # 2
Principles of Wireless Networks

Problem 1 (Coherent capacity: Symmetric assumption)

Consider the angular representation $\mathbf{H^a}$ of the MIMO channel $\mathbf{H} = \mathbf{U}_r\mathbf{H^a}\mathbf{U}_t^*$. We statistically model $\mathbf{H^a}$ as a $M_r \times M_t$ random matrix with independent columns, the distribution of whose entries is jointly symmetric with respect to zero.

(a). Starting with the expression for the capacity of the MIMO channel with receiver CSI, show that

$$C = \max_{\mathbf{K}_x:\mathrm{Tr}(\mathbf{K}_x)\leq P} \mathbb{E}\Big[\log\det\Big(I_{M_r} + \frac{1}{N_0}\mathbf{H^a}\mathbf{U}_t^*\mathbf{K}_x\mathbf{U}_t\mathbf{H^{a*}}\Big)\Big]$$

(b). Show that we can restrict the input covariance $\mathbf{K}_x$ to be of the following structure

$$\mathbf{K}_x = \mathbf{U}_t\Lambda\mathbf{U}_t^*$$

where Λ is a diagonal matrix with non negative entries that sum to P. Hint: Start with defining a diagonal matrix Π_i with -1 in the i^{th} position and 1 in the remaining positions.

Problem 2 (Universal code design criterion for the MISO channel)

Consider the slow fading fading MISO channel with M_t transmit antennas and a single receive antenna, *i.e.,*

$$y[m] \quad = \quad \mathbf{h}^*\mathbf{x}[m] + z[m] \tag{1}$$

where $\mathbf{h} = \left(h_1,\ldots,h_{M_t}\right)^\top$ with $\mathbf{h} \sim \mathbb{C}\mathcal{N}(0,\mathbf{I})$ while $z[m] \sim \mathbb{C}\mathcal{N}(0,1)$ is i.i.d. over time. The pairwise error probability (of confusing codeword $\mathbf{X}_A$ with $\mathbf{X}_B$) conditioned on a specific channel realization is given by

$$\Pr(\mathbf{X}_A \to \mathbf{X}_B|\mathbf{h}) = Q\big(\frac{\|\mathbf{h}^*(\mathbf{X}_A - \mathbf{X}_B)\|}{\sqrt{2}}\big)$$

The worst case error probability over all channels not in outage is given by

$$\max_{\mathbf{h}:\|\mathbf{h}\|^2 \geq \frac{M_t(2^R-1)}{SNR}} Q\big(\frac{\|\mathbf{h}^*(\mathbf{X}_A - \mathbf{X}_B)\|}{\sqrt{2}}\big)$$

(a). Show that this probability can be explicitly written as

$$Q\Big(\sqrt{\frac{1}{2}\lambda_1^2 M_t(2^R-1)}\Big)$$

where λ_1 is the smallest singular value of the normalized codeword difference matrix $\frac{1}{\sqrt{SNR}}(\mathbf{X}_A - \mathbf{X}_B)$.

(b). Let $\hat{\lambda}_1$ be the smallest singular value of $(X_A - X_B)$. What is the minimum value of $\hat{\lambda}_1$ so that the worst case error still goes down exponentially with the SNR?

Problem 3 (Diversity-Multiplexing tradeoff - Alamouti scheme over the $2 \times M_r$ MIMO)

[Exercise 9.4 from the same text.]
Consider using the Alamouti scheme over a $2 \times M_r$ i.i.d. Rayleigh fading MIMO channel, given as

$$\mathbf{y}[t] = \mathbf{H}\mathbf{x}[t] + \mathbf{w}[t].$$

The transmit codeword matrix spans two symbol times $t = 1, 2$:

$$\begin{bmatrix} u_1 & -u_2^* \\ u_2 & u_1^* \end{bmatrix}.$$

(a). With this input to the MIMO channel, show that we can write the output over the two time symbols as

$$\begin{bmatrix} \mathbf{y}[1] \\ (\mathbf{y}[2]^*)^t \end{bmatrix} = \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 \\ (\mathbf{h}_2^*)^t & -(\mathbf{h}_1^*)^t \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} \mathbf{w}[1] \\ (\mathbf{w}[2]^*)^t \end{bmatrix}. \quad (2)$$

Here we have denoted the two columns of $\mathbf{H}$ by $\mathbf{h}_1$ and $\mathbf{h}_2$.

(b). Observing that the two columns of the effective channel matrix in (2) are orthogonal, show that e can extract simple sufficient statistics for the data symbols u_1 and u_2 as

$$v_i = \| \mathbf{H} \| u_i + w_i, \qquad i = 1, 2. \quad (3)$$

Here $\| \mathbf{H} \|^2$ denotes $\| \mathbf{h}_1 \|^2 + \| \mathbf{h}_2 \|^2$, and the additive noises w_1 and w_2 are i.i.d. $\mathbb{C}\mathcal{N}(0,1)$.

(c). Conclude that the maximum diversity gain seen by either stream (u_1 or u_2) at a multiplexing rate of r per stream is $2M_t(1-r)$.

Problem 4 (Diversity-Multiplexing tradeoff over L parallel channels)

[Exercise 9.2 from the text "Fundamentals of Wireless Communications" by Tse-Viswanath.]
Consider the repetition scheme where the same codeword is transmitted over the L i.i.d. Rayleigh sub-channels of a parallel channel. Show that the largest diversity gain this scheme can achieve at a multiplexing rate of r per sub-channel is $L(1-Lr)$.

Problem 5 (Diversity-Multiplexing tradeoff - V-Blast with nulling)

[Exercise 9.5 from the same text.]
Consider the V-BLAST architecture with a bank of decorrelators for the $M_t \times M_r$ i.i.d. Rayleigh fading MIMO channel with $M_r \geq M_t$. Show that the effective channel seen by each stream is a scalar fading channel with distribution $\chi^2_{2(M_r - M_t + 1)}$. Conclude that the diversity gain with a multiplexing gain of r is $(M_r - M_t + 1)(1 - r/M_t)$.

Problem 6 (Diversity multiplexing tradeoff using superposition codes)

Consider a scalar block fading channel,

$$y^{(b)}(k) \;=\; h^{(b)} x^{(b)}(k) + z^{(b)}(k) \qquad k = 0, \ldots, T-1 \quad (4)$$

where the channel $h^{(b)}$ remains constant for T time units. Let $z^{(b)}(k) \sim \mathbb{C}\mathcal{N}(0,1)$ be i.i.d. Gaussian noise and we have a transmit power constraint $\mathbb{E}|x(k)|^2 \leq SNR$. The channel is assumed to be Rayleigh fading *i.e.* $h^{(b)} \sim \mathbb{C}\mathcal{N}(0,1)$ and varies independently from block to block. Assume that the coding interval is T *i.e.,* we do not code across fading blocks. The transmission codebook is Gaussian and we transmit at a rate $R(SNR) = r\log(SNR)$. We have a sequence of codebooks for each SNR level and we are interested in characterizing the diversity multiplexing tradeoff for some strategies over this channel. Note that we assume the receiver knows $\{h^{(b)}\}$ accurately whereas the transmitter does not have access to it.

(a). Characterize the diversity multiplexing tradeoff for the scalar channel given in (4).

Hint: You do not need lengthy calculations, just prove the outage diversity order and state the achievable coding strategy

(b). Now consider a superposition scheme which uses

$$x^{(b)}(k) = x_H^{(b)}(k) + x_L^{(b)}(k) \tag{5}$$

where $\{x_H^{(b)}(k)\}$ and $\{x_L^{(b)}(k)\}$ are designed for two message sets M_H and M_L. These messages can be delivered to the receiver though the same (common) channel $\{h^{(b)}\}$. Therefore (4) is modified to,

$$y^{(b)}(k) = h^{(b)}x_H^{(b)}(k) + h^{(b)}x_L^{(b)}(k) + z^{(b)}(k) \qquad k = 0, \ldots, T-1 \tag{6}$$

If we use Gaussian codebooks for both message sets and allocate power SNR_H to message M_H and SNR_L to message M_L, we see that $SNR_H + SNR_L \leq SNR$ is needed. Let

$$SNR_H \doteq SNR \quad , \quad SNR_L \doteq SNR^{1-\beta} \tag{7}$$

If we want to decode both M_H and M_L, characterize the outage diversity order versus multiplexing tradeoff, in terms of the multiplexing rates of M_H and M_L, *i.e.*, $R_H(SNR) = r_H \log(SNR)$, $R_L(SNR) = r_L \log(SNR)$,

$$\tilde{d} = \lim_{SNR \to \infty} \frac{\log P_{out}(M_H, M_L, SNR)}{\log SNR} \tag{8}$$

and $P_{out}(M_H, M_L, SNR)$ is the outage probability for the joint decoder.

(c). Suppose we use a successive decoder, where we decode M_H first considering M_L as part of the noise. Find an expression for $P_{out}(M_H, SNR)$ the outage probability for such a system.

(d). Characterize the outage diversity order for the system in (c) in terms of β, r_H, *i.e.* find,

$$d_H = \lim_{SNR \to \infty} \frac{\log P_{out}(M_H, SNR)}{\log SNR} \tag{9}$$

(e). Is there a choice of β, $\beta \neq 1$, such that d_H is the same as in (a)?