Solutions: Homework Set # 1
Principles of Wireless Networks

Problem 1 (Antenna spacing and angular representation)

(a). The bin k (k ranging from 0 to $n_r - 1$) corresponds to the non resolvable paths that are within a solid angle of $\pm\frac{1}{2L_r}$ of a path that subtends an angle ϕ_k with respect to the array. The ϕ_k satisfies

$$\cos\phi_k = \frac{k}{L_r}(\text{mod}\frac{1}{\Delta_r}) \\ = \frac{k}{3}(\text{mod}2)$$

Note that $\Delta_r = \frac{L_r}{n_r} = 0.5$. We can explicitly solve the above equation for each ϕ_k to get $\phi_0 = \cos^{-1} 0 = 90°, 270°$, $\phi_1 = \cos^{-1}\frac{1}{3}, 360° - \cos^{-1}\frac{1}{3}$, $\phi_2 = \cos^{-1}\frac{2}{3}, 360° - \cos^{-1}\frac{2}{3}$, $\phi_3 = \cos^{-1} 1 = 0$ and $\phi_3 = \cos^{-1} -1 = 180°$, $\phi_4 = \cos^{-1} -\frac{2}{3}, 360° - \cos^{-1} -\frac{2}{3}$ and $\phi_5 = \cos^{-1} -\frac{1}{3}, 360° - \cos^{-1} -\frac{1}{3}$. Note that for the case of $k = 3$, there are two solutions for $\cos\phi_3$, namely 1 and -1.

The angular width of the 0^{th} bin is given by $2*2*(90° - \cos^{-1}\frac{1}{6}) = 38.38°$, the 1^{st} and 5^{th} bin each is given by $2*2*(\cos^{-1}\frac{1}{6} - \cos^{-1}\frac{1}{3}) = 39.50°$, the 2^{nd} and 4^{th} bin each is given by $2*2*(2\cos^{-1}\frac{1}{3} - \cos^{-1}\frac{2}{3} - \cos^{-1}\frac{1}{6}) = 49.85°$ and the 5^{th} bin is given by $2*2*(2\cos^{-1}\frac{2}{3} + \cos^{-1}\frac{1}{6} - 2\cos^{-1}\frac{1}{3}) = 142.9°$.

The width in radians increases as ϕ decreases from 90° to 0°. This is because of the relatively faster changes in the value of $\cos\phi$ for higher values of ϕ and correspondingly, the relatively slower changes in the value of $\cos\phi$ for lower values of ϕ.

(b). For the case where $\Delta_r = \frac{3}{5}$, the ϕ_k satisfies

$$\cos\phi_k = \frac{k}{3}(\text{mod}\frac{5}{3})$$

The solutions for the ϕ_k are given by $\phi_0 = \cos^{-1} 0 = 90°, 270°$, $\phi_1 = \cos^{-1}\frac{1}{3}, 360° - \cos^{-1}\frac{1}{3}$, $\phi_2 = \cos^{-1}\frac{2}{3}, 360° - \cos^{-1}\frac{2}{3}, \cos^{-1}(-1) = 180°$, $\phi_3 = \cos^{-1}(1), \cos^{-1}(-\frac{2}{3}), 360° - \cos^{-1} -\frac{2}{3}$ and $\phi_4 = \cos^{-1} -\frac{1}{3}, 360° - \cos^{-1} -\frac{1}{3}$.
The angular width of the 0^{th} bin is given by $2*2*(90° - \cos^{-1}\frac{1}{6}) = 38.38°$, the 1^{st} and 4^{th} bin each is given by $2*2*(\cos^{-1}\frac{1}{6} - \cos^{-1}\frac{1}{3}) = 39.50°$, the 2^{nd} and 3^{rd} bin each is given by $2*(2\cos^{-1}\frac{1}{3} - \cos^{-1}\frac{1}{6}) = 60.65° = 121.3°$.

For the case where $\Delta_r = \frac{3}{2}$, the ϕ_k satisfies

$$\cos\phi_k = \frac{k}{3}(\text{mod}\frac{2}{3})$$

The solutions for the ϕ_k are given by $\phi_0 = 90°, 270°, \cos^{-1}\frac{2}{3}, 360° - \cos^{-1}\frac{2}{3}, \cos^{-1}\frac{-2}{3}, 360° - \cos^{-1}\frac{-2}{3}$, and $\phi_1 = 0°, 180°, \cos^{-1}\frac{1}{3}, 360° - \cos^{-1}\frac{1}{3}, \cos^{-1}\frac{-1}{3}, 360° - \cos^{-1}\frac{-1}{3}$.
The angular width of the 0^{th} bin is given by $38.38 + 2*49.85 = 138.08°$ and that of the 1^{st} bin is given by $360 - 138.08 = 221.92°$.

(c). For the case where $\Delta_r = \frac{3}{10}$, the ϕ_k satisfies

$$\cos\phi_k = \frac{k}{3}(\mathrm{mod}\frac{10}{3})$$

The solutions for the ϕ_k are given by $\phi_0 = \cos^{-1} 0 = 90°, 270°$, $\phi_1 = \cos^{-1}\frac{1}{3}, 360° - \cos^{-1}\frac{1}{3}$, $\phi_2 = \cos^{-1}\frac{2}{3}, 360° - \cos^{-1}\frac{2}{3}$, $\phi_3 = \cos^{-1} 1 = 0$, $\phi_7 = \phi_3 = \cos^{-1} -1 = 180°$, $\phi_8 = \cos^{-1} -\frac{2}{3}, 360° - \cos^{-1} -\frac{2}{3}$ and $\phi_9 = \cos^{-1} -\frac{1}{3}, 360° - \cos^{-1} -\frac{1}{3}$. Note there is no solution for $k = 4, 5, 6$.
The angular width of the 0^{th} bin is given by $2*2*(90° - \cos^{-1}\frac{1}{6}) = 38.38°$, the 1^{st} and 9^{th} bin each is given by $2*2*(\cos^{-1}\frac{1}{6} - \cos^{-1}\frac{1}{3}) = 39.50°$, the 2^{nd} and 8^{th} bin each is given by $2*2*(2\cos^{-1}\frac{1}{3} - \cos^{-1}\frac{2}{3} - \cos^{-1}\frac{1}{6}) = 49.85°$ and the 3^{rd} and 7^{th} bin each, is given by $2*(2\cos^{-1}\frac{2}{3} + \cos^{-1}\frac{1}{6} - 2\cos^{-1}\frac{1}{3}) = 71.45°$.

Bin numbers $4, 5$ and 6 do not correspond to any angular direction and are hence empty.

(d),(e). For the four cases, we identify the receiver bins in which the paths fall into.

Path No.(i)	Solid angle at Rx ant.($\Omega_{r,i}$)	Case $n_r = 6$ (a)	Case $n_r = 2$ (b)	Case $n_r = 10$ (c)
1	-0.40	4	0	8
2	0.0	0	0	0
3	0.80	2	0	2
4	0.10	0	0	0
5	-0.90	3	1	7
6	0.10	0	0	0
7	-0.40	4	0	8
8	0.20	1	1	1
9	-0.60	4	0	8
10	0.70	2	0	2
11	-0.45	4	0	8
12	-0.80	4	0	8

Since there is a single transmit antenna, there is only physical bin at the transmitter into which all the paths fall. The resultant H^a matrix is a $n_r \times 1$ matrix with coefficients given by Eq(7.72). Notice that the paths belonging to a bin are aggregated together.

Observe that in the densely spaced case, the bins $l = 3, 4, 5, 6, 9$ are empty and therefore the corresponding coefficients H_l^a will be negligible. When the antenna spacing is increased, the bins $l = 3, 9$ are empty and the corresponding coefficients H_l^a will be negligible. The effect of increasing the length is to reduce the beam width of the angular lobes and thereby paths which previously fall into the same physical bin now fall into different physical bins. This increases the number of degrees of freedom in the channel.

Problem 2 (Coherent capacity: Fast Fading SIMO channel)

(a). (i) follows from the independence of $\underline{x}$ and $\underline{\mathbf{h}}$. (ii) follows since removing terms from the conditioning cannot reduce the differential entropy and because $\mathbf{y}[i]$ is independent of everything else given $(\mathbf{h}[i], x[i])$.

$$
\begin{aligned}
R &= \frac{1}{n} I(\underline{x}; \underline{\mathbf{y}}, \underline{\mathbf{h}}) \\
&= \frac{1}{n} I(\underline{x}; \underline{\mathbf{h}}) + \frac{1}{n} I(\underline{x}; \underline{\mathbf{y}} | \underline{\mathbf{h}}) \\
&\stackrel{(i)}{=} \frac{1}{n} I(\underline{x}; \underline{\mathbf{y}} | \underline{\mathbf{h}}) \\
&= \frac{1}{n} h(\underline{\mathbf{y}} | \underline{\mathbf{h}}) - \frac{1}{n} h(\underline{\mathbf{y}} | \underline{x}, \underline{\mathbf{h}}) \\
&= \frac{1}{n} \sum_{i=1}^{n} h(\mathbf{y}[i] | \mathbf{y}[1], \ldots, \mathbf{y}[i-1], \underline{\mathbf{h}}) - h(\mathbf{y}[i] | \mathbf{y}[1], \ldots, \mathbf{y}[i-1], \underline{x}, \underline{\mathbf{h}}) \\
&\stackrel{(ii)}{\leq} \frac{1}{n} \sum_{i=1}^{n} h(\mathbf{y}[i] | \mathbf{h}[i]) - h(\mathbf{y}[i] | \mathbf{h}[i], x[i]) \\
&= \frac{1}{n} \sum_{i=1}^{n} h(\mathbf{y}[i] | \mathbf{h}[i]) - h(\mathbf{z}[i]) \\
&\stackrel{(iii)}{\leq} h(\mathbf{y} | \mathbf{h}) - h(\mathbf{z}) \\
&= I(x; \mathbf{y} | \mathbf{h})
\end{aligned}
$$

We explain the last inequality. Notice that $h(\mathbf{y}[i]|\mathbf{h}[i])$ is a concave function of the distribution $p_i(\cdot)$ on the letter $x[i]$. Therefore, $\frac{1}{n}\sum_{i=1}^{n} h\big(p_i(\cdot)\big) \leq h\big(\frac{1}{n}\sum_{i=1}^{n} p_i(\cdot)\big)$. The convex combination of the distributions $\{p_i(\cdot)\}$ is the distribution with respect to which the differential entropy is calculated. This rate is achievable using random coding sampling each letter of the codeword from the above distribution.

(b). (i) follows since the circularly symmetric Gaussian has the maximum differential entropy among random variables with a given second moment. (ii) follows since $\det|I+AB| = \det|I+BA|$. and (iii) follows since the $\log(\cdot)$ is an increasing function of the input power which is constrained by P.

(c). Since the information $x[m]$ is multiplied by the vector $\mathbf{h}[m]$, there is no information carried by any of the vectors in a direction orthogonal to $\mathbf{h}[m]$. The operation of premultiplying by $\frac{\mathbf{h}^*[m]}{\|\mathbf{h}[m]\|}$ extracts the component of the received vector in the direction of $\mathbf{h}[m]$ and annuls the components in the orthogonal direction. Since there is no loss of information, $\tilde{y}[m]$ is a sufficient statistic.

(d). The equivalent scalar channel is

$$
\tilde{y}[m] = \|\mathbf{h}[m]\| x[m] + \tilde{z}[m]
$$

where $\tilde{z}[m] \sim \mathbb{C}\eta(0, \sigma^2)$. The capacity of this channel is also $\mathbb{E}_{\mathbf{h}} \log\left(1 + \frac{P}{\sigma^2}\|\mathbf{h}\|^2\right)$

Problem 3 (Coherent capacity: Fast fading MISO channel)

(a). (i) follows since the circularly symmetric Gaussian has the maximum differential entropy among random variables with a given second moment. (ii) follows since $K_{\mathbf{x}}$ is Hermitian and therefore has a SVD into $\mathbf{U}^*\Lambda_{\mathbf{x}}\mathbf{U}$. (iii) follows since $\mathbf{U}\mathbf{h}$ has the same distribution as $\mathbf{h}$.

(b). (i) follows since $\Pi^T\mathbf{h}$ has the same distribution as $\mathbf{h}$. (ii) follows since $\log(\cdot)$ is a concave function. The entries in $\tilde{\Lambda}_{\mathbf{x}}$ are equal. Since $\log(\cdot)$ is an increasing function the maximum is attained when the diagonal entries are all $\frac{P}{t}$ and (iii) follows.

Problem 4 (Beamforming)

(a). From the Cauchy-Schwarz inequality, the received SNR

$$\frac{||\mathbf{h}^*[m]\mathbf{x}[m]||^2}{\sigma^2} \leq \frac{||\mathbf{h}[m]||^2\ ||\mathbf{x}[m]||^2}{\sigma^2}$$

If we need to transmit $\tilde{x}$ the strategy $\mathbf{x}[m] = \frac{\mathbf{h}[m]}{||\mathbf{h}[m]||}\tilde{x}[m]$ achieves the upper bound in the equation above. Hence this strategy maximizes the received SNR.

(b). The equivalent scalar channel between $\tilde{x}[m]$ and $y[m]$ is given by

$$y[m] = \|\mathbf{h}[m]\|\tilde{x}[m] + z[m]$$

The optimal strategy is given by waterfilling across time as

$$\mathbf{x}[m] = \frac{\mathbf{h}[m]}{||\mathbf{h}[m]||}\tilde{x}[m]$$

with $|\tilde{x}[m]|^2 = P_m$ such that

$$P_m = \left(\alpha - \frac{\sigma^2}{||\mathbf{h}[m]||^2}\right)^+$$

with α chosen such that

$$\sum_{m=1}^{T} P_m = TP$$

Problem 5 (Degrees of Freedom)

Let $\mathbf{h}_1, \ldots, \mathbf{h}_k$ be k columns, each generated from $\mathcal{C}^{n_r}\eta(0,I)$. Let $\mathcal{A} \subset \mathcal{C}^{n_r}$ be the l dimensional subspace ($l \leq k$) spanned by $\mathbf{h}_1, \ldots, \mathbf{h}_k$. Let $\mathbf{u}_1, \ldots \mathbf{u}_l$ be the l basis vectors of $\mathcal{A}$. Extend the basis vectors to bases for $\mathcal{C}^{n_r}$. Let $\mathbf{U}$ be the matrix with columns $\mathbf{u}_1, \ldots \mathbf{u}_{n_r}$.

We can express $\mathbf{h}_{k+1} = \sum_1^{n_r} \tilde{h}_{k+1}(i)\mathbf{u}_i$. The vector $\tilde{\mathbf{h}}_{k+1}$ is distributed as $U^*\mathbf{h}_{k+1}$, which in turn is distributed as $\mathbf{h}_{k+1}$, i.e, as $\mathcal{C}^{n_r}\eta(0,I)$. The probability that $\mathbf{h}_{k+1} \in \mathcal{A}$ is the probability that $\tilde{h}_{k+1}(i) = 0$ for $i = l+1, \ldots, n_r$. Since $\tilde{h}_{k+1}(i)$ is distributed as a circularly symmetric Gaussian variable, and is in particular continuous, this probability is zero. For $k < n_r$, therefore, the probability that $\mathbf{h}_{k+1}$ is in the linear span of $\{\mathbf{h}_1, \ldots, \mathbf{h}_k\}$ is zero. Applying this argument for $k = 1, \ldots, n_t - 1$ gives the required result.

Problem 6 (MMSE Successive Interference Cancellation)

(a). Assume that the first $k-1$ streams have already been decoded without error and subtracted out.

$$\begin{aligned}
\Rightarrow \mathbf{y}^{(k)}[m] &= \mathbf{h}_k x_k[m] + \sum_{i=k+1}^{M_t} \mathbf{h}_i x_i[m] + \mathbf{z}[\mathbf{m}] \\
&= \mathbf{h}_k x_k[m] + \mathbf{z}^{(k)}[m] \\
\mathbf{K}_k &= \mathbb{E}(\mathbf{z}^{(k)}[m]\mathbf{z}^{(k)*}[m]) \\
&= N_0\mathbf{I}_{M_r} + \sum_{i=k+1}^{M_t} \mathbf{h}_i\mathbf{h}_i^* P_i \\
&= \mathbf{K}_{\mathbf{z}[\mathbf{m}]}
\end{aligned}$$

Processing the output (after subtracting the decoded streams) with the MMSE filter results in a scalar channel with SNR $P_k \mathbf{h}_k^* \mathbf{K}_k^{-1} \mathbf{h}_k$. The maximum rate at which stream k can reliably carry information is therefore

$$R_k = \log(1 + P_k \mathbf{h}_k^* (N_0 \mathbf{I}_{M_r} + \sum_{i=k+1}^{M_t} \mathbf{h}_i \mathbf{h}_i^* P_i)^{-1} \mathbf{h}_k)$$

(b).

$$\begin{aligned} R_{M_t} &= \log\left(1 + \frac{P_{M_t}}{N_0} \mathbf{h}_{M_t}^* \mathbf{h}_{M_t}\right) \\ &= \log\left(\mathbf{I}_{M_r} + \frac{P_{M_t}}{N_0} \mathbf{h}_{M_t} \mathbf{h}_{M_t}^*\right) \; [using\ det(I+AB) = det(I+BA)] \end{aligned}$$

Similarly

$$R_{M_t} = \log|\mathbf{I}_{M_r} + P_{M_t-1}(N_0 \mathbf{I}_{M_r} + P_{M_t} \mathbf{h}_{M_t} \mathbf{h}_{M_t}^*)^{-1} \mathbf{h}_{M_t-1} \mathbf{h}_{M_t-1}^*|$$

Therefore,

$$\begin{aligned} R_{M_t} + R_{M_t-1} &= \log\left(1 + \frac{P_{M_t}}{N_0} \mathbf{h}_{M_t}^* \mathbf{h}_{M_t}\right) + \log(1 + P_{M_t-1} \mathbf{h}_{M_t-1}^* (N_0 \mathbf{I}_{M_r} + P_{M_t} \mathbf{h}_{M_t} \mathbf{h}_{M_t}^*)^{-1} \mathbf{h}_{M_t-1}) \\ &= \log\left|\left(\mathbf{I}_{M_r} + \frac{P_{M_t}}{N_0} \mathbf{h}_{M_t} \mathbf{h}_{M_t}^*\right)(\mathbf{I}_{M_r} + P_{M_t-1}(N_0 \mathbf{I}_{M_r} + P_{M_t} \mathbf{h}_{M_t} \mathbf{h}_{M_t}^*)^{-1} \mathbf{h}_{M_t-1} \mathbf{h}_{M_t-1}^*)\right| \\ &= \log\left|\mathbf{I}_{M_r} + \frac{P_{M_t}}{N_0} \mathbf{h}_{M_t} \mathbf{h}_{M_t}^* + \frac{P_{M_t-1}}{N_0}(N_0 \mathbf{I}_{M_r} + P_{M_t} \mathbf{h}_{M_t} \mathbf{h}_{M_t}^*)(N_0 \mathbf{I}_{M_r} + P_{M_t} \mathbf{h}_{M_t} \mathbf{h}_{M_t}^*)^{-1} \mathbf{h}_{M_t-1} \mathbf{h}_{M_t-1}^*\right| \\ &= \log\left|\mathbf{I}_{M_r} + \frac{P_{M_t}}{N_0} \mathbf{h}_{M_t} \mathbf{h}_{M_t}^* + \frac{P_{M_t-1}}{N_0} \mathbf{h}_{M_t-1} \mathbf{h}_{M_t-1}^*\right| \end{aligned}$$

Continuing in a similar manner we get

$$\sum_{k=1}^{M_t} R_k = \log\left|\mathbf{I}_{M_r} + \frac{1}{N_0} \sum_{i=1}^{M_t} P_i \mathbf{h}_i \mathbf{h}_i^*\right|$$

(c). If $P_i = \frac{P}{M_t}$, the above expression becomes,

$$\begin{aligned} \sum_{k=1}^{M_t} R_k &= \log\left|\mathbf{I}_{M_r} + \frac{P}{N_0 M_t} \sum_{i=1}^{M_t} \mathbf{h}_i \mathbf{h}_i^*\right| \\ &= \log\left|\mathbf{I}_{M_r} + \frac{SNR}{M_t} \sum_{i=1}^{M_t} \mathbf{h}_i \mathbf{h}_i^*\right| \\ &= \log\left|\mathbf{I}_{M_r} + \frac{SNR}{M_t} \sum_{i=1}^{M_t} \mathbf{H}\mathbf{H}^*\right| \end{aligned}$$