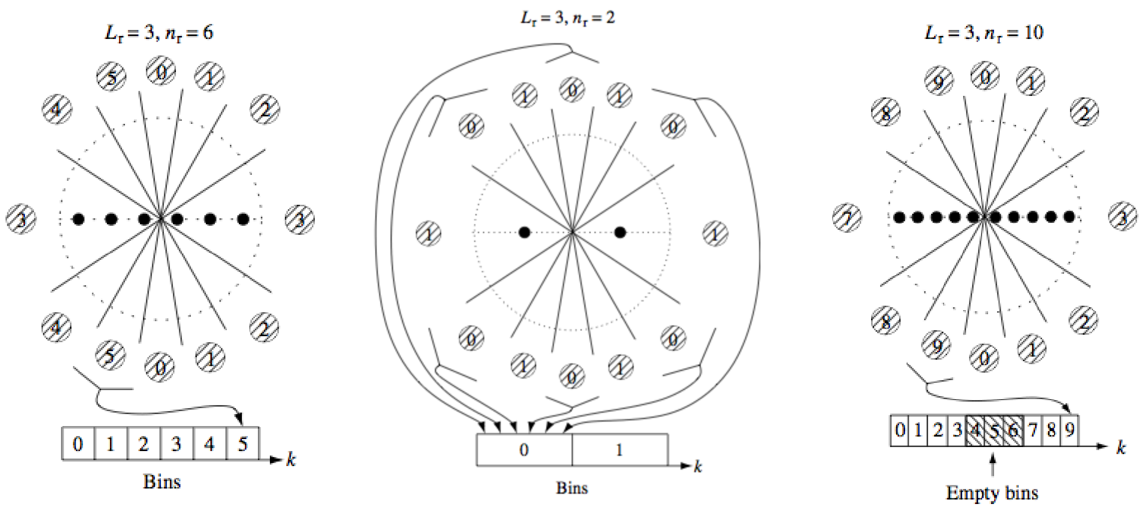
Homework Set # 1
Principles of Wireless Networks

Problem 1 (Antenna spacing and angular representation)

[Exercise 7.5 from the text “Fundamentals of Wireless Communications” by Tse-Viswanath.]
In this question we study the role of antenna spacing in the angular representation of the MIMO channel.

(a). Consider the critically spaced antenna array in Fig 1(a); there are six bins, each one corresponding to a specific physical angular window. All of these angular windows have the same width as measured in solid angle. Compute the angular window width in radians for each of the bins $\mathcal{T}_\ell$, with $\ll= 0,\ldots,5$. Argue that the width in radians increases as we move from the line perpendicular to the antenna array to one that is parallel to it.
Hint: Refer to equation (7.64) in text by Tse-Viswanath.



(a) Critically spaced antennas at half of the wavelength

(b) Sparsely spaced antennas

(c) Densely spaced antennas

Figure 1: Mapping from angular windows to bins.

(b). Now consider the sparsely spaced antenna array in Fig 1(b). Justify the depicted mapping from the angular windows to the bins $\mathcal{T}_\ell$ and evaluate the angular window width in radians for each of the bins $\mathcal{T}_\ell$ with $\ell = 0,1$. (The angular window width of a bin $\mathcal{T}_\ell$ is the sum of the all the angular windows that correspond to the bin $\mathcal{T}_\ell$.)

(c). Justify the depiction of the mapping from angular windows to the bins $\mathcal{T}_\ell$ in the densely spaced antenna array of Fig. 1(c). Also evaluate the angular width of each bin in radians.

(d). The table below gives the attenuation, the angle with respect to the transmit antenna array and the angle with respect to the receive antenna array for the 12 available paths between the transmitter and the receiver antenna locations. Let there is only one transmit antenna ($n_t = 1$) and we have $L_t = 0.5$. Using this information, compute the entries in the angular representation $\mathbf{H^a}$ of the channel matrix for the three cases corresponding to parts (a), (b), and (c).

Path No.(i)	Attenuation (a_i^b)	Solid angle at Tx ant. ($\Omega_{t,i}$)	Solid angle at Rx ant. ($\Omega_{r,i}$)
1	0.29 + 0.36j	0.50	-0.40
2	0.11 + 1.21j	-0.90	0.0
3	1.38 + 5.60j	0.40	0.80
4	0.40 + 0.07j	-0.20	0.10
5	0.91 + 1.01j	0.80	-0.90
6	2.76 + 2.11j	0.70	0.10
7	0.67 + 0.02j	0.40	-0.40
8	3.21 + 2.11j	-0.10	0.20
9	1.16 + 5.57j	0.30	-0.60
10	2.89 + 1.71j	0.20	0.70
11	0.42 + 0.18j	0.70	-0.45
12	1.15 + 0.82j	0.10	-0.80

(e). Let the length of the receive antenna be increased to $L_r = 5$ while keeping the number of antennas $n_r = 10$. How does the angular representation $\mathbf{H^a}$ of the channel matrix compare with the angular representation in the densely spaced case? What does this tell about the effect of increasing the antenna length while keeping the number of antenna elements the same?

Problem 2 (Coherent capacity: Fast Fading SIMO channel)

Consider a fast fading fading SIMO channel with a single transmit antenna and M_r receive antennas. The channel model is described by

$$\mathbf{y}[m] \quad = \quad \mathbf{h}[m]x[m] + \mathbf{z}[m] \tag{1}$$

where $\mathbf{h}[m] = \left(h_1[m], \ldots, h_{M_r}[m]\right)^T$ and $h_l[m]$ is the channel gain at the m^{th} time instant from the transmit antenna to the l^{th} receive antenna and is distributed as $h_l[m] \sim \mathbb{C}\eta(0,1)$ while $\mathbf{z}[m] \sim \mathbb{C}\eta(0, \sigma^2\mathbf{I})$. The channel gains and the noise are i.i.d over time. The transmitter satisfies a long term power constraint $\lim_{T\to\infty} \frac{1}{T}\sum_{m=1}^{T} \|x[m]\|^2 \leq P$. Assume that the channel realization is known at the receiver but not at the transmitter.

(a). An achievable rate of communication is given by $\frac{1}{n}I(\underline{x}; \underline{\mathbf{y}}, \underline{\mathbf{h}})$ where n is the block length used for communication and the underlined quantities represent blocks, i.e., $\underline{x} = (x[1], x[2], \ldots, x[n])$, $\underline{\mathbf{y}} = (\mathbf{y}[1], \mathbf{y}[2], \ldots, \mathbf{y}[n])$ and $\underline{\mathbf{h}} = (\mathbf{h}[1], \mathbf{h}[2], \ldots, \mathbf{h}[n])$. The mutual information is computed for some distribution $p(x^n)$ of the input block. Justify the following set of inequalities,*i.e.,* prove (i),(ii),(iii).

$$\begin{aligned}
R &= \frac{1}{n}I(\underline{x}; \underline{\mathbf{y}}, \underline{\mathbf{h}}) \\
&\overset{(i)}{=} \frac{1}{n}I(\underline{x}; \underline{\mathbf{y}}|\underline{\mathbf{h}}) \\
&\overset{(ii)}{\leq} \frac{1}{n}\sum_{i=1}^{n} h(\mathbf{y}[i]|\mathbf{h}[i]) - h(\mathbf{y}[i]|\mathbf{h}[i], x[i]) \\
&\overset{(iii)}{=} \mathbb{E}_{\mathbf{h}} I(x; \mathbf{y}|\mathbf{h})
\end{aligned}$$

In the last equality, what is the distribution on x with respect to which the mutual information is evaluated? Conclude that there is no loss in the achievable rate when restricting $p(\underline{x})$ to be of the form $p(\underline{x}) = \Pi_{i=1}^{n} p(x[i])$ such that $\mathbb{E}\|x\|^2 \leq P$.

(b). The capacity of the channel is $\max_{p(x)} \mathbb{E}_{\mathbf{h}} I(x;\mathbf{y}|\mathbf{h})$ subject to $\mathbb{E}\|x\|^2 \le P$. Starting with a distribution $p(x)$, justify the following set of inequalities,*i.e.,* prove (i),(ii),(iii).

$$
\begin{aligned}
\mathbb{E}_{\mathbf{h}} I(x;\mathbf{y}|\mathbf{h}) &= \mathbb{E}_{\mathbf{h}}[h(\mathbf{y}|\mathbf{h}) - h(\mathbf{y}|x,\mathbf{h})] \\
&\overset{(i)}{\le} \mathbb{E}_{\mathbf{h}} \log \left| \mathbf{I} + \frac{1}{\sigma^2}\mathbf{h}\mathbb{E}[xx^*]\mathbf{h}^* \right| \\
&\overset{(ii)}{=} \mathbb{E}_{\mathbf{h}} \log \left(1 + \frac{\mathbb{E}[xx^*]}{\sigma^2}\|\mathbf{h}\|^2 \right) \\
&\overset{(iii)}{\le} \mathbb{E}_{\mathbf{h}} \log \left(1 + \frac{P}{\sigma^2}\|\mathbf{h}\|^2 \right)
\end{aligned}
$$

(c). Justify that a sufficient statistic for detecting $x[m]$ from $\mathbf{y}[m]$ is

$$
\tilde{y}[m] = \frac{\mathbf{h}^*[m]}{\|\mathbf{h}[m]\|}\mathbf{y}[m]
$$

(d). What is the equivalent scalar channel obtained after processing the output in the above manner? What is the capacity of this channel?

Problem 3 (Coherent capacity: Fast fading MISO channel)

Consider the fast fading fading MISO channel with M_t transmit antennas and a single receive antenna, *i.e.,*

$$
y[m] \quad = \quad \mathbf{h}^*[m]\mathbf{x}[m] + z[m] \tag{2}
$$

where $\mathbf{h}[m] = \left(h_1[m],\ldots,h_{M_t}[m]\right)^T$ and $h_l[m]$ is the channel gain at the m^{th} time instant from the l^{th} transmit antenna to the receive antenna and is distributed as $h_l[m] \sim \mathbb{C}\eta(0,1)$ while $z[m] \sim \mathbb{C}\eta(0,1)$. The channel gain and the noise are i.i.d over time. Assume that the channel realization is known at the receiver but not at the transmitter.

(a). The capacity of the channel is $\max_{p(\mathbf{x})} I(\mathbf{x};y|\mathbf{h})$ subject to $\mathbb{E}\|x\|^2 \le P$. Let the distribution on $\mathbf{x}$ be $p(\mathbf{x})$ and let $\mathbf{K_x}$ represent the covariance matrix of $\mathbf{x}$. Justify the following set of inequalities,*i.e.,* prove (i),(ii),(iii).

$$
\begin{aligned}
\mathbb{E}_{\mathbf{h}} I(\mathbf{x};y|\mathbf{h}) &= \mathbb{E}_{\mathbf{h}} h(y|\mathbf{h}) - h(y|\mathbf{h},\mathbf{x}) \\
&\overset{(i)}{\le} \mathbb{E}_{\mathbf{h}} \log \left(1 + \frac{1}{\sigma^2}\mathbf{h}^* K_{\mathbf{x}}\mathbf{h} \right) \\
&\overset{(ii)}{=} \mathbb{E}_{\mathbf{h}} \log \left(1 + \frac{1}{\sigma^2}\mathbf{h}^*\mathbf{U}^*\Lambda_{\mathbf{x}}\mathbf{U}\mathbf{h} \right) \\
&\overset{(iii)}{=} \mathbb{E}_{\mathbf{h}} \log \left(1 + \frac{1}{\sigma^2}\mathbf{h}^*\Lambda_{\mathbf{x}}\mathbf{h} \right)
\end{aligned}
$$

Conclude that there is no loss in achievable rate when considering distributions $p(\mathbf{x})$ with a diagonal covariance matrix.

(b). Given any such diagonal matrix $\Lambda_{\mathbf{x}}$ and any permutation matrix Π, consider $\Lambda_{\mathbf{x}}^{\Pi} = \Pi\Lambda_{\mathbf{x}}\Pi^T$. Define $\tilde{\Lambda}_{\mathbf{x}} = \frac{1}{t!}\sum_{\Pi}\Lambda_{\mathbf{x}}^{\Pi}$. Justify the following set of inequalities,*i.e.,* prove (i),(ii),(iii).

$$
\begin{aligned}
\mathbb{E}_{\mathbf{h}} \log \left(1 + \frac{1}{\sigma^2}\mathbf{h}^*\Lambda_{\mathbf{x}}\mathbf{h} \right) &\overset{(i)}{=} \frac{1}{t!}\sum_{\Pi}\mathbb{E}_{\mathbf{h}} \log \left(1 + \frac{1}{\sigma^2}\mathbf{h}^*\Lambda_{\mathbf{x}}^{\Pi}\mathbf{h} \right) \\
&\overset{(ii)}{\le} \mathbb{E}_{\mathbf{h}} \log \left(1 + \frac{1}{\sigma^2}\mathbf{h}^*\tilde{\Lambda}_{\mathbf{x}}\mathbf{h} \right) \\
&\overset{(iii)}{=} \mathbb{E}_{\mathbf{h}} \log \left(1 + \frac{P}{\sigma^2 t}\|\mathbf{h}\|^2 \right)
\end{aligned}
$$

Problem 4 (Beamforming)

Consider again the fast fading MISO channel model with M_t transmit antennas and a single receive antenna, with channel realization known at *both* the transmitter and the receiver.

Let the signal to be sent be $\tilde{x}[m]$ and the transmit beamforming strategy sends the following signal on the M_t transmit antennas,

$$\mathbf{x}[m] = \frac{\mathbf{h}[m]}{||\mathbf{h}[m]||}\tilde{x}[m]$$

Show that this strategy maximizes received *SNR*.

Problem 5 (Degrees of Freedom)

[Exercises 8.12 and 8.13 from the text "Fundamentals of Wireless Communications" by Tse-Viswanath.]

Suppose $\mathbf{H}$ of size $n_r \times n_t$ (with $n_t < n_r$) has i.i.d. $\mathcal{CN}(0,1)$ entries and denote the columns of $\mathbf{H}$ by $\mathbf{h}_1, \ldots, \mathbf{h}_{n_t}$.

(a). Show that the probability that the columns are linearly dependent is zero. Hence, conclude that the probability that rank of $\mathbf{H}$ is strictly smaller that n_t is zero.

(b). Show that the dimension of the subspace spanned by the vectors $\mathbf{h}_1 \ldots, \mathbf{h}_{k-1}, \mathbf{h}_{k+1}, \ldots, \mathbf{h}_{n_t}$ is $n_t - 1$ with probability 1. Hence, conclude that the dimension of the subspace V_k, orthogonal to this one, has dimension $n_r - n_t + 1$ with probability 1.

Problem 6 (MMSE Successive Interference Cancellation)

Consider a MIMO system with M_t transmit and M_r receive antennas with power constraint P. The received vector at symbol time m is,

$$\mathbf{y}[m] = \sum_{i=1}^{M_t} \mathbf{h}_i x_i[m] + \mathbf{z}[m]$$

where $\mathbf{h}_1, \ldots, \mathbf{h}_{M_t}$ are the columns of $\mathbf{H}$, the elements of $\mathbf{H}$ are i.i.d. Gaussian and $(x_i[m])$ are the independent data streams transmitted on the i^{th} antenna. Let us order the data streams as $1, \ldots, M_t$ and consider a sequence of linear MMSE receivers followed by successive cancellation to decode the data streams.

(a). Show that using the MMSE-SIC receiver, the rate at which stream k can be reliably decoded is given by

$$R_k = \log\left|1 + P_k \mathbf{h}_k^* \left(N_0 \mathbf{I}_{M_r} + \sum_{i=k+1}^{M_t} P_i \mathbf{h}_i \mathbf{h}_i^*\right)^{-1} \mathbf{h}_k\right|$$

(b). Show that the sum-rate is given by

$$\sum_{k=1}^{M_t} R_k = \log\left|\mathbf{I}_{M_r} + \frac{1}{N_0}\sum_{i=1}^{M_t} P_i \mathbf{h}_i \mathbf{h}_i^*\right|$$

(c). Setting $P_i = \frac{P}{M_t}$, show that:

$$\sum_{k=1}^{M_t} R_k \quad = \quad \log \left| \mathbf{I} + \frac{SNR}{M_t} \mathbf{H}\mathbf{H}^* \right|$$

Conclude that linear MMSE followed by successive cancellation of independent equal power data streams, one on each of the transmit antennas, achieves the capacity of the MIMO channel.