Information Theory and Coding — EPFL Winter Semester 2008/2009
Prof. Suhas Diggavi — Handout # 11, Thursday, 23 October, 2008

Solutions: Homework Set # 3

Problem 1

(a) $S_n = \frac{X_1+X_2+....+X_n}{n}$

$$\mathbb{E}\{S_n\} = \frac{n\mu}{n} = \mu$$

$$\begin{aligned}
\mathbb{E}\big\{(S_n-\mu)^2\big\} &= \mathbb{E}\left\{\left(\frac{(X_1-\mu)+(X_2-\mu)+\cdots+(X_n-\mu)}{n}\right)^2\right\} \\
&= \mathbb{E}\left\{\frac{(x_1-\mu)^2+(x_2-\mu)^2+....+(x_n-\mu)^2}{n^2}\right\} \\
&= \frac{\sigma^2}{n}
\end{aligned}$$

Note that $\mathbb{E}\{(X_i-\mu)(X_j-\mu)\} = \mathbb{E}\{X_iX_j\} - \mu\mathbb{E}\{X_i\} - \mu\mathbb{E}\{X_j\} + \mu^2 = 0$

(b) Define $Y_n = (S_n-\mu)^2$.

$$\Rightarrow Pr[(S_n-\mu)^2 > \epsilon^2] = Pr[Y_n > \epsilon^2] \leq \frac{\mathbb{E}\{Y_n\}}{\epsilon^2}$$
$$\Rightarrow Pr[(S_n-\mu)^2 > \epsilon^2] \leq \frac{\mathbb{E}\big\{(S_n-\mu)^2\big\}}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2}$$

which tends to zero as n $\to \infty$.

Problem 2

(a) The fake coin can be at n possible positions. Thus, there are n possible combinations.

(b) By each use of a balance, the one third of the coins which contains the fake coin can be found.(Partition the coins into three groups and compare two groups which have the same number of elements. If they are equal in weight, the coin is in the other group. If they are different in weight, the fake coin is in the group on the lighter arm). Thus, $\log_2 3$ bits of information is gained at each time using the balance.

Another way to answer this question is to consider the possible outputs of a balance:
- either the arms are equal in weight,
- or the left arm is lighter,
- or the right arm is lighter.
So $\log_2 3$ bits of information can be revealed if you make occurence of each of the outputs equal. (as an example of a poor method, if you group the coins into 2 groups, you are in fact setting the probability of occuring equal arms to zero and you are not gaining all the

information that the balance can reveal.)

The minimum number of time to use the balance is

$$\frac{\log_2 n}{\log_2 3} = \log_3 n$$

(c) A detector has two possible outcomes; either it shows true or false. This gives $\log_2 2$ bits of information. Thus $\log_2 n$ using the detector is the minimum number of times that a detector should be used to find the fake coin.

(d) This time we have $2n$ possible combinations since at each position the fake coin can be either lighter or heavier.

(e) The amount of information revealed at each time using the balance is again $\log_2 3$. However, this time the number of possible combinations is $2n$. Thus, the number required weighting the balance is

$$\frac{\log_2 2n}{\log_2 3}.$$

(f) Using the same argument,

$$\frac{\log_2 2n}{\log_2 2} = \log_2 n + 1.$$

However, note that the term 1 which is being added to $\log_2 n$ is the 1 bit information to distinguish L and H and this is not achievable by a detector and is not required anyway (The fake coin is to be found and not more). So $\log_2 n$ is the minimum number of weighings by a detector to detect the fake coin.[This argument is not true for the balance as whenever a balance finds the fake coin, it finds out if it is H or L as well. Thus the '1' bit distinguishing L and H of the coin cannot be regarded as 'extra' or 'unwanted bit of information' to be ignored).]

The other point of view is to classify the $2n$ combinations to n combinations where the L or H does not matter and argue that $\log_2 n$ weighings via the detector finds the required class and this is all we are asked to find.

Problem 3

(a) The situation is equivalent to having objects picked from a mixed distribution $(\frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16})$. The optimal sequence of questions corresponds to the Huffman code for this source:

$$\begin{aligned} A &\to 0 \\ B &\to 10 \\ C &\to 1100 \\ D &\to 1101 \\ E &\to 1110 \\ F &\to 1111 \end{aligned}$$

The 1^{st} question is: is the object A?
The 2^{nd} question is: is the object B?
The 3^{rd} question is: is the object C or D?

The 4^{th} question is: is the object C or E?

The average number of questions is $L_Q = 2$.

(b)

$$\begin{aligned}
A &\to 0 \\
B &\to 10 \\
C &\to 110 \\
D &\to 111 \\
E &\to - \\
F &\to -
\end{aligned}$$

The average number of questions is $L_p = 1.75$. (Similarly $L_q = 1.75$)

(c) If the genie is always giving the information, then we have

$$L_G = \frac{1}{2}L_P + \frac{1}{2}L_Q = 1.75$$

The information given by the genie is

$$L_Q - L_G = 0.25 \;\; bits$$

(d) If we use the genie every time, then the average number of questions asked is

$$L_{G^*} = L_G + 1 = 2.75 \;\; bits$$

This is clearly sub-optimal. Instead, we can proceed as in part (a), and then use the genie (if needed) instead of the 3^{rd} question, since he won't be needed before that. (Indeed, if asked at that point, the genie is equivalent to the 3^{rd} question).

Problem 4

$$\begin{aligned}
V_n &= \prod_i^n X_i \\
l &= V_n^{1/n} \qquad (1)
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} V_n^{1/n} &= e^{\log(\lim_{n\to\infty} V_n^{1/n})} \\
&= e^{\lim_{n\to\infty} \frac{1}{n}\log V_n} \\
&= e^{\mathbb{E}\{\log X_i\}} \\
&= e^{-1}
\end{aligned}$$

because

$$\begin{aligned}
\mathbb{E}\{\log X_i\} &= \int_0^1 \log x dx \\
&= \lim_{\epsilon\to 0}[x\log x - x]_\epsilon^1 \\
&= \lim_{\epsilon\to 0}(-1 - \epsilon\log\epsilon + \epsilon) = -1
\end{aligned}$$

$$
\begin{aligned}
(\mathbb{E}\{V_n\})^{1/n} &= (\mathbb{E}\left\{\prod_i^n X_i\right\})^{1/n} \\
&= (\prod_i^n \mathbb{E}\{X_i\})^{1/n} \\
&= (\frac{1}{2^n})^{1/n} = \frac{1}{2}
\end{aligned}
$$

This clarifies how the expected edge length does not capture the idea of the volume of the box.