More Exercises

Problem 1 (How Many Fingers Has a Martian?)

Consider the source $S = \{S_1, \cdots, S_m\}$ where each S_i has a probability of p_i. The S_i's are encoded into the strings from a D-symbol output alphabet in a uniquely decodable manner. If $m = 6$ and the codeword lengths are $(l_1, l_2, \cdots, l_6) = (1, 1, 2, 3, 2, 3)$, find a good lower bound on D.

Problem 2 (Huffman Codes)

Consider a random variable X that takes six values $\{A, B, C, D, E, F\}$ with probabilities $\{0.5, 0.25, 0.1, 0.05, 0.05, 0.05\}$.

(a) Construct a binary Huffman code for this random variable. What is its average length?

(b) Construct a quaternary Huffman code for this random variable (i.e., construct a code over an alphabet of four symbols ($\{a, b, c, d\}$ for example)). What is the average length of this code?

(c) One way to construct a binary code for the random variable is to start with a quaternary code and convert the symbols into binary using the mapping

$$\begin{aligned} a &\to 00 \\ b &\to 01 \\ c &\to 10 \\ d &\to 11 \end{aligned}$$

What is the average length of the binary code which is constructed in this method for the random variable above?

(d) For any random variable X, let L_H be the average length of the binary Huffman code for this random variable, and let L_{QB} be the average length of the binary code which is constructed from the Huffman quaternary code (using the method of (c)). Show that

$$L_H \leq L_{QB} \leq L_H + 2.$$

(e) Give an example where the Binary code constructed from an optimal Huffman quaternary code is also the optimal binary code and conclude that the lower bound is tight.

(f) prove that $L_{QB} \leq L_H + 1$ and show that it is tight with an example. So the upper bound of (e) is in fact not tight.

Problem 3 (Huffman 20 questions)

Consider a set of n objects. Let $X_i = 1$ or 0 according as the i^{th} object is good or defective. Let $X_1, X_2, \cdots X_n$ be independent with $Pr\{X_i = 1\} = p_i$ (We assume that $p_1 > p_2 > \cdots > p_n > \frac{1}{2}$). We are asked to determine the set of all defective objects and any yes-no question is admissible.

(a) Give a good lower bound on the minimum average number of questions required.

(b) If the longest sequence of questions is required by nature's answers to our questions, what is the last question we should ask? What two sets are we distinguishing with this question (Assume a compact (minimum average length) sequence of question)?

(c) Give an upper bound (within one question) on the minimum average number of questions required.

Problem 4 (Concavity of $I(X;Y)$ in $p(x)$)

In this exercise, you will prove that $I(X;Y)$ is concave in $p(x)$ when $p(y|x)$ is fixed.
Let X_1 and X_2 be two random variables with distributions p_1 and p_2 respectively, taking values in a set A. Assume that X is a random variable defined as follows:

$$X = \begin{cases} X_1 & Z = 0 \\ X_2 & Z = 1 \end{cases}$$

where the random variable Z takes values 0 and 1 with probabilities λ and $1 - \lambda$ respectively.

(a) What is the distribution of X?

(b) Compare $I(X;Y|Z)$ and $I(X;Y)$.

(c) Show that $I(X;Y|Z) = \lambda I(X_1;Y) + (1 - \lambda) I(X_2;Y)$.

(d) Conclude that $I(X;Y)$ is concave in $p(x)$.

Problem 5 (Mixing increases entropy)

(a) Show that the entropy of the probability distribution $(p_1, \ldots, p_i, \cdots, p_j, \cdots, p_m)$ is less than the entropy of the distribution $(p_1, \cdots, \frac{p_i+p_j}{2}, \cdots, \frac{p_i+p_j}{2}, \cdots, p_m)$.

(b) Show than in general any averaging on the p_i's (which makes the distribution more uniform) increases the entropy.

Problem 6 (AEP)

Let $X_1, X_2, \cdots$ be independent, identically distributed random variables drawn according to the probability mass fucntion $p(x)$, $x \in \{1, 2, \cdots, m\}$. Thus, $p(x_1, x_2, \ldots, x_n) = \prod_{i=1}^n p(x_i)$ and $-\frac{1}{n} \log p(X_1, X_2, \ldots, X_n) \to H(X)$ in probability. Let $q(x_1, x_2, \cdots, x_n) = \prod_{i=1}^n q(x_i)$, where q is another probability mass function on $\{1, 2, \cdots, m\}$.

(a) Evaluate $\lim -\frac{1}{n} \log q(X_1, X_2, \cdots, X_n)$, where $X_1, X_2 \cdots$ are i.i.d. and from the distribution $p(x)$.

(b) Evaluate the limit of the log likelihood ratio $\frac{1}{n} \log \frac{q(X_1, \cdots, X_n)}{p(X_1, \cdots, X_n)}$ when $X_1, X_2, \cdots$ are i.i.d. and from the distribution $p(x)$. Conclude that the odds favoring q are exponentially small when p is true.