PROBLEM 1. The output of the system with the input $\delta[n]$ is $\delta[n]$ and with $\delta[n+1]$ is $-\delta[n+1]$, which is not a shifted version of $\delta[n]$. Thus the system is not time invariant.

PROBLEM 2. We know that the sequences $e^{j\omega n}$ are eigenfunctions of LTI systems, that is, $\mathcal{H}\{e^{j\omega n}\} = H(\omega)e^{j\omega n}$. Therefore, an LTI system cannot alter the frequency of its input. The system given in the problem outputs a sequence with frequency $\pm\frac{\pi}{2}$ when the input frequency is $\pm\frac{2\pi}{5}$. Thus the system cannot be LTI.

PROBLEM 3. 1. For $M = 2$, $x[n]$ is plotted in Figure 1.

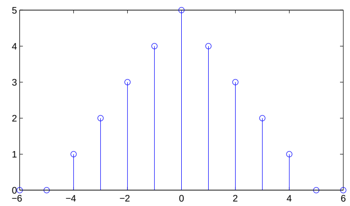


Figure 1:

2. We know that $H(e^{j\omega}) = \frac{\sin(\omega(M+\frac{1}{2}))}{\sin(\omega/2)}$ (cf. Prandoni, Vetterli, p. 74). Therefore,

$$X(e^{j\omega}) = H(e^{j\omega})H(e^{j\omega}) = \left(\frac{\sin(\omega(M+\frac{1}{2}))}{\sin(\omega/2)}\right)^2. \quad (1)$$

$|X(e^{j\omega})|$ is plotted in Figure 2.

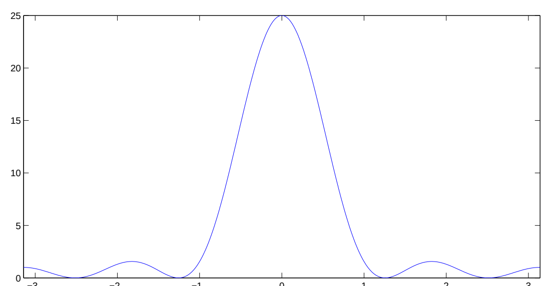


Figure 2:

3. From equation (1), it is easy to see that the 'main lobe' of $X(e^{j\omega})$ gets narrower as $M$ grows.

4. $y[n] = x[n] * h[n]$ is plotted in Figure 3. Since $x[n]$ is piecewise linear and since convolution with $h[n]$ is equivalent to a discrete-time integration, $y[n]$ will be quadratice for general $M$.

5. $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = H(e^{j\omega})^3$. $|Y(e^{j\omega})|$ is plotted in Figure 4.

PROBLEM 4. The response of the system can be found as

$$\begin{aligned}(X(z) + Y(z)H_2(z))\,H_1(z) &= Y(z)\\ \Rightarrow H(z) &= \frac{Y(z)}{X(z)}\\ &= \frac{H_1(z)}{1 + H_1(z)H_2(z)}\end{aligned}$$

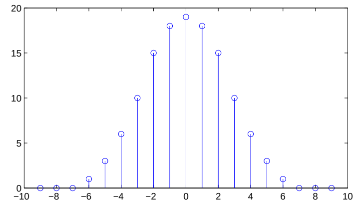


Figure 3:

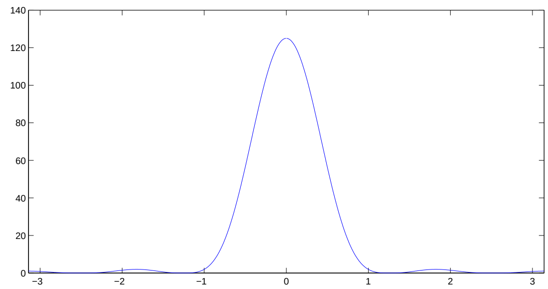


Figure 4:

where

$$H_1(z) = 1 \quad \text{and} \quad H_2(z) = \frac{-\frac{1}{2}z^{-1}}{1 + \frac{1}{2}z^{-1}}.$$

The systems represented by $h_1[n]$ and $h_2[n]$ are stable since $\sum_{n=-\infty}^{\infty} |h_1[n]| = 1$ and $\sum_{n=-\infty}^{\infty} |h_2[n]| = 1$.

Then, the overall response is:

$$\begin{aligned} H(z) &= \frac{1}{1 + \frac{-\frac{1}{2}z^{-1}}{1+\frac{1}{2}z^{-1}}} \\ &= 1 + \frac{1}{2}z^{-1} \\ &\Rightarrow h[n] = \delta[n] + \frac{1}{2}\delta[n-1]. \end{aligned}$$

The overall system is causal and stable. If we take $h_2[n] = (-\frac{1}{2})^{n-1}u[n-1]$ which also represents a causal and a stable system with $\sum_{n=-\infty}^{\infty} |h_2[n]| = 2$, the response $H_2(z)$ is

$$H_2(z) = \frac{z^{-1}}{1 + \frac{1}{2}z^{-1}}.$$

Then the overall response of the system is:

$$\begin{aligned} H(z) &= \frac{1}{1 + \frac{z^{-1}}{1+\frac{1}{2}z^{-1}}} \\ &= \frac{1 + \frac{1}{2}z^{-1}}{1 + \frac{3}{2}z^{-1}} \\ &= 1 - \frac{z^{-1}}{1 + \frac{3}{2}z^{-1}}. \end{aligned}$$

There is a pole at $z = -3/2$ and we know that the system is causal. Then, the ROC is $|z| > 3/2$ and the unit circle is not included in the unit circle so the system is not stable. The overall impulse response is $h[n] = \delta[n] + (-\frac{3}{2})^{n-1}u[n-1]$.

PROBLEM 5. $h_1[n] = \delta[n]$, $h_2[n] = (-\frac{1}{2})^n u[n-1]$. We have

$$Y(z) = (X(z) - Y(z)H_2(z))H_1(z),$$

where $H_1(z) = \mathcal{Z}\{h_1\} = 1$ and $H_2(z) = \mathcal{Z}\{h_2\} = \frac{-\frac{1}{2}z^{-1}}{1+\frac{1}{2}z^{-1}}$. Therefore the transfer function of the system is

$$\frac{1}{1+H_2(z)} = 1 + \frac{1}{2}z^{-1}.$$

This system is causal by construction. Also, the above transfer function has no poles (except at zero), therefore the system is stable.

The transfer function of the system, with $h_2[n] = (-\frac{1}{2})^{n-1}u[n-1]$, is

$$\frac{1}{1+H_2(z)} = \frac{1}{1+\frac{2\cdot\frac{1}{2}z^{-1}}{1+\frac{1}{2}z^{-1}}} = \frac{1+\frac{1}{2}z^{-1}}{1+\frac{3}{2}z^{-1}}.$$

This time, there is a pole at $-\frac{3}{2}$. Since the system is causal, ROC$= \{z : |z| > 3/2\}$, which does not contain the unit circle. Therefore, the system is not stable.

PROBLEM 6. The DTFT of the input signal $x[n]$ is given by

$$X(e^{j\omega}) = \pi \sum_{l=-\infty}^{\infty} \delta(\omega - 0.6\pi - 2\pi l) + \delta(\omega + 0.6\pi - 2\pi l) + 3e^{-j5\omega} + 4\pi \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m).$$

Note that $X(e^{j\omega})$ is always periodic with $2\pi$. We can write $X(e^{j\omega})$ for one period as:

$$X(e^{j\omega}) = \pi\left(\delta(\omega - 0.6\pi) + \delta(\omega + 0.6\pi)\right) + 3e^{-j5\omega} + 4\pi\delta(\omega).$$

We know that $W(e^{j\omega}) = X(e^{j\omega})H_1(e^{j\omega})$, then for one period $W(e^{j\omega})$ is

$$W(e^{j\omega}) = \begin{cases} 3e^{-j5\omega} + 4\pi\delta(\omega) & \text{if } |\omega| \le 0.5\pi \\ 0 & \text{else} \end{cases}.$$

The signal $w[n]$ can be computed as:

$$\begin{aligned}\frac{1}{2\pi}\int W(e^{j\omega})e^{j\omega n}d\omega &= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3e^{-j5\omega} + 4\pi\delta(\omega)e^{j\omega n}d\omega \\ &= \frac{3}{2\pi}\left(\frac{e^{j\frac{\pi}{2}(n-5)} - e^{-j\frac{\pi}{2}(n-5)}}{j(n-5)}\right) + 2 \\ &= \frac{3\sin(\frac{\pi}{2}(n-5))}{\pi(n-5)} + 2.\end{aligned}$$

Then $y[n] = \omega[n] - \omega[n-1]$ is given by

$$y[n] = \frac{3\sin(\frac{\pi}{2}(n-5))}{\pi(n-5)} - \frac{3\sin(\frac{\pi}{2}(n-6))}{\pi(n-6)}.$$