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Handout 9 Signal Processing for Communications
Solutions to Homework 3 April 5, 2009

PROBLEM 1. 1.

$$\begin{aligned}
\sum_{n=0}^{N-1} x^*[-n \mod N] e^{-j\frac{2\pi}{N}nk} &= \sum_{n=0}^{N-1} x^*[N-n] e^{-j\frac{2\pi}{N}nk} \\
&= (\sum_{n=0}^{N-1} x[N-n] e^{j\frac{2\pi}{N}nk})^* \\
&= (\sum_{n=0}^{N-1} x[n] e^{j\frac{2\pi}{N}(N+n)k})^* \\
&= (\sum_{n=0}^{N-1} x[n] e^{j\frac{2\pi}{N}nk})^* \\
&= X[k]^*
\end{aligned}$$

2.

$$\begin{aligned}
\sum_{n=0}^{N-1} X[n] e^{-j\frac{2\pi}{N}nk} &= \sum_{n=0}^{N-1} N \sum_{m=0}^{N-1} x[m] e^{-j\frac{2\pi}{N}mn} e^{-j\frac{2\pi}{N}nk} \\
&= \sum_{m=0}^{N-1} x[m] \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}n(m+k)} \\
&= \sum_{m=0}^{N-1} x[m] \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}n(m+k)} \\
&= \sum_{m=0}^{N-1} x[m] N\delta(-k+Nl) \quad l \in \mathbb{Z} \\
&= Nx[N-k] = Nx[-k \mod N]
\end{aligned}$$

3. When $x[n]$ is real (i.e. $x[n] = x^*[n]$)

$$\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x^*[n] e^{-j\frac{2\pi}{N}nk} \\
&= (\sum_{n=0}^{N-1} x[n] e^{j\frac{2\pi}{N}nk})^* \\
&= (\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}n(-k)})^* \\
&= X[-k \mod N]^*
\end{aligned}$$

$\Re\{X[k]\} + \Im\{X[k]\} = \Re\{X[-k \mod N]\} - j\Im\{X[-k \mod N]\}$
$\rightarrow \Re\{X[k]\} = \Re\{X[-k \mod N]\}$ and $\Im\{X[k]\} = -\Im\{X[-k \mod N]\}$
This implies that the real part of $X[k]$ is symmetric and the imaginary part of $X[k]$ is anti-symmetric.

4. Since $x[n]$ is real $x[n] = x^*[n]$. Then

$$x_e[n] = \frac{1}{2}(x[n] + x^*[-n \mod N]) \longleftrightarrow \frac{1}{2}(X[k] + X^*[k]) = \Re\{X[k]\}$$

$$x_o[n] = \frac{1}{2}(x[n] - x^*[-n \mod N]) \longleftrightarrow \frac{1}{2}(X[k] - X^*[k]) = j\Im\{X[k]\}$$

Problem 2. Denote the DFT of $x_1[n]x_2[n]$ by X. Then we have

$$\begin{aligned} X_1[k] \circledast X_2[k] =& \frac{1}{N}\sum_{l=0}^{N-1} X_1[l]X_2[k-l \mod N] \\ =& \frac{1}{N}\sum_{l=0}^{N-1}\sum_{m=0}^{N-1} x_1[m]e^{-j\frac{2\pi}{N}ml}\sum_{r=0}^{N-1} x_2[r]e^{-j\frac{2\pi}{N}r(k-l \mod N)} \\ =& \frac{1}{N}\sum_{m=0}^{N-1}\sum_{r=0}^{N-1} x_1[m]x_2[r]\sum_{l=0}^{N-1} e^{-j\frac{2\pi}{N}(ml+r(k-l \mod N))} \end{aligned}$$

Note that

$$\sum_{l=0}^{N-1} e^{-j\frac{2\pi}{N}(ml+r(k-l \mod N))} = \begin{cases} 0 & \text{if } m \neq r \\ Ne^{-j\frac{2\pi}{N}mk} & \text{if } m = r \end{cases}.$$

Therefore,

$$X_1[k] \circledast X_2[k] = \sum_{m=0}^{N-1} x_1[m]x_2[m]e^{-j\frac{2\pi}{N}mk} = X[k].$$

Problem 3. The dirac delta is $\delta(t)$ is defined through the equality

$$\int_{-\infty}^{\infty} \delta(t)f(t)dt = f(0).$$

By a change of variable one can easily show that

$$\int_{-\infty}^{\infty} \delta(t-a)f(t)dt = f(a),$$

$$\delta(ax-b) = \frac{\delta(x-\frac{b}{a})}{|a|}.$$

Using these we obtain,

1.
$$\int_{-\infty}^{\infty} \delta(-t)f(t^2)dt = \int_{-\infty}^{\infty} \delta(t)f(t^2)dt = f(0).$$

2.

$$\int_{-\infty}^{\infty} \delta(2t-5)f(3t+2)dt = \frac{1}{2}\int_{-\infty}^{\infty} \delta(t-\frac{5}{2})f(3t+2)dt = \frac{1}{2}f(3\frac{5}{2}+2) = \frac{1}{2}f(\frac{19}{2}).$$

3.

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(-\frac{3t}{4}+1)f(-(\frac{t}{2})^2)dt &= \frac{4}{3}\int_{-\infty}^{\infty} \delta(t-\frac{4}{3})f(-(\frac{t}{2})^2)dt = \frac{4}{3}f(-(\frac{4/3}{2})^2)\\ &= \frac{4}{3}f(-(\frac{2}{3})^2) = \frac{4}{3}f(-\frac{4}{9}).\end{aligned}$$

4.

$$\int_{-\infty}^{\infty} \delta(\frac{2t}{3}+4)(f(t))^3dt = \frac{3}{2}\int_{-\infty}^{\infty} \delta(t+6)(f(t))^3dt = \frac{3}{2}(f(-6))^3.$$

5.

$$\begin{aligned}&\int_{-\infty}^{\infty} \delta(-4t+5)(2f(2t)^2+f(2t-1)+2)dt\\ &= \frac{1}{4}\int_{-\infty}^{\infty} \delta(t-\frac{5}{4})(2f(2t)^2+f(2t-1)+2)dt\\ &= \frac{1}{4}(2f(2\frac{5}{4})^2+f(2\frac{5}{4}-1)+2)\\ &= \frac{1}{2}f(\frac{5}{2})^2+\frac{1}{4}f(\frac{3}{2})+\frac{1}{2}.\end{aligned}$$

PROBLEM 4. 1. Let $\mathcal{V}$ denote the set of all ordered n-tuples. Let $\mathbf{0}=$ be the all-zero n-tuple, and let $\mathbf{e}_i$ be the n-tuple with a 1 at the ith position, and 0's in the rest of the positions.

- Commutativity and associativity of addition is clear.
- Distributivity of scalar multiplication can be checked easily.
- $\mathbf{0}$ is the null vector:

$$\mathbf{0}+[a_1,\ldots,a_n] = [a_1,\ldots,a_n]+\mathbf{0} = [a_1,\ldots,a_n].$$

- The additive inverse of $[a_1,\ldots,a_n]$ is $[-a_1,\ldots,-a_n]$:

$$[a_1,\ldots,a_n]+[-a_1,\ldots,-a_n] = \mathbf{0}.$$

- 1 is the identity element for scalar multiplication

$$1\cdot[a_1,\ldots,a_n] = [a_1,\ldots,a_n]\cdot 1 = [a_1,\ldots,a_n].$$

Therefore $\mathcal{V}$ is a vector space. Clearly, every n-tuple $[a_1,\ldots,a_n]$ can be written as

$$\sum_{k=1}^{n} a_i\mathbf{e}_i.$$

Therefore $\{\mathbf{e}_i\}_{i=1}^n$ is a basis for $\mathcal{V}$. Thus, the dimension of the vector space is n.

2. Let $z(x) = 0 = 0\sin(x) + 0\cos(x)$ be the zero function. Then it can easily be checked that the set of functions $y(x) = a\sin(x) + b\cos(x)$ for any a, b satisfies the properties of a vector space, its zero element being $z(x)$, and the scalar 1 being the identity element for multiplication.

   The functions $e_1(x) = \sin(x)$ and $e_2(x) = \cos(x)$ form a basis for this vector space. Therefore the dimension is 2.

3. Denote the eight corners of a cube by $c_1 = (0,0,0)$, $c_2 = (0,0,1)$, $c_3 = (0,1,0)$, $c_4 = (1,0,0)$, $c_5 = (0,1,1)$, $c_6 = (1,1,0)$, $c_7 = (1,0,1)$, $c_8 = (1,1,1)$. Then the diagonal vectors are $\mathbf{d}_1 = (1,1,1)$, $\mathbf{d}_2 = (1,1,-1)$, $\mathbf{d}_3 = (1,-1,1)$, $\mathbf{d}_4 = (-1,1,1)$. Then

$$\mathbf{d}_1^T\mathbf{d}_2 = 1 + 1 - 1 = 1 \neq 0$$

   Therefore the diagonals are not orthogonal.

4.

$$\delta[n] = u[n] - u[n-1],$$

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k].$$

5. Define

$$f_e(t) = \frac{f(t) + f(-t)}{2}, \quad f_o(t) = \frac{f(t) - f(-t)}{2}.$$

   It is easy to see that $f_e(t) = f_e(-t)$ and $f_o(t) = -f_o(t)$, and also that $f_e(t) + f_o(t) = f(t)$.

PROBLEM 5. 1. $X(e^{j\omega}) = \sum_n x[n]e^{-j\omega n} = \sum_{n=-3}^{2} ne^{-j\omega n} = \cos(\omega) + 2\cos(2\omega) + e^{-j3\omega}$.

2.

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} n\frac{1}{3}^{|n|} u[-n-2]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{-2} n\frac{1}{3}^{|n|} e^{-j\omega n} \\ &= -\sum_{n=2}^{\infty} n\frac{1}{3}e^{j\omega}{}^{n} \\ &= -\frac{1}{3}e^{j\omega}\sum_{n=2}^{\infty} n\frac{1}{3}^{n-1} \end{aligned} \quad (1)$$

Now note that for $|\alpha| < 1$ we have

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$$

Taking derivatives with respect to α we get

$$\sum_{n=0}^{\infty} n\alpha^{n-1} = \frac{1}{(1-\alpha)^2}$$

Plugging this in (1) we get

$$X(e^{j\omega}) = -\frac{1}{3}e^{j\omega}\left(\frac{1}{(1-\frac{1}{3}e^{j\omega})^2} - 1\right).$$

3.

$$\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} 2^n \sin(\frac{\pi}{4}n)u[-n]e^{-j\omega n} \\
&= \sum_{n=-\infty}^{0} 2^n \sin(\frac{\pi}{4}n)e^{-j\omega n} \\
&= -\sum_{n=0}^{\infty}(\frac{1}{2}e^{j\omega})^n \sin(\frac{\pi}{4}n) \\
&= -\sum_{n=0}^{\infty}(\frac{1}{2}e^{j\omega})^n \frac{e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}}{2j} \\
&= -\sum_{n=0}^{\infty}(\frac{1}{2}e^{j\omega})^n \frac{e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}}{2j} \\
&= -\frac{1}{2j}\sum_{n=0}^{\infty}(\frac{1}{2}e^{j\omega+\frac{\pi}{4}})^n e^{j\frac{\pi}{4}n} + \frac{1}{2j}\sum_{n=0}^{\infty}(\frac{1}{2}e^{j\omega-\frac{\pi}{4}})^n \\
&= -\frac{1}{2j(1-\frac{1}{2}e^{j\omega+\frac{\pi}{4}})} + \frac{1}{2j(1-\frac{1}{2}e^{j\omega-\frac{\pi}{4}})}
\end{aligned}$$

In the above, the third equality follows from $\sin(t) = -\sin(-t)$.

4.

$$\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty}\left(\sin\left(\frac{\pi}{2}n + \frac{\pi}{6}\right) + \cos(n)\right)e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty}\sin\left(\frac{\pi}{2}n + \frac{\pi}{6}\right)e^{-j\omega n} + \sum_{n=-\infty}^{\infty}\cos(n)e^{-j\omega n} \\
&= -\frac{j}{2}\left[e^{j\frac{\pi}{6}}\tilde{\delta}(\omega - \frac{\pi}{2}) - e^{-j\frac{\pi}{6}}\tilde{\delta}(\omega + \frac{\pi}{2})\right] \\
&\quad + \frac{1}{2}\left[\tilde{\delta}(\omega - 1) - \tilde{\delta}(\omega + 1)\right].
\end{aligned}$$

5.

$$\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty}\cos\left(\frac{\pi}{4}n\right)\sin\left(\frac{2\pi}{5}n\right)e^{-j\omega n} \\
&= \left(\sum_{n=-\infty}^{\infty}\cos\left(\frac{\pi}{4}n\right)e^{-j\omega n}\right) * \left(\sum_{n=-\infty}^{\infty}\sin\left(\frac{2\pi}{5}n\right)e^{-j\omega n}\right) \\
&= \left(\frac{1}{2}\left[\tilde{\delta}(\omega - \frac{\pi}{4}) + \tilde{\delta}(\omega + \frac{\pi}{4})\right]\right) * \left(\frac{-j}{2}\left[\tilde{\delta}(\omega - \frac{2\pi}{5}) - \tilde{\delta}(\omega + \frac{2\pi}{5})\right]\right) \\
&= \frac{-j}{4}\left[-\tilde{\delta}\left(\omega - \frac{3\pi}{20}\right) + \tilde{\delta}\left(\omega - \frac{13\pi}{20}\right) + \tilde{\delta}\left(\omega + \frac{3\pi}{20}\right) - \tilde{\delta}\left(\omega + \frac{13\pi}{20}\right)\right].
\end{aligned}$$