ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 4 Signal Processing for Communications

Solutions to Homework 2 March 9, 2009

PROBLEM 1.

1. We have

$$\begin{aligned} X^*[-k \mod N] &= X^*[N-k] \\ &= \sum_{n=0}^{N-1} \left(g_1^*[n] - jg_2^*[n]\right) e^{j\left(\frac{2\pi}{N}\right)(N-k)n} \\ &= \sum_{n=0}^{N-1} \left(g_1[n] - jg_2[n]\right) e^{j\left(\frac{2\pi}{N}\right)(N-k)n} \\ &= \sum_{n=0}^{N-1} \left(g_1[n] - jg_2[n]\right) e^{-j\left(\frac{2\pi}{N}\right)kn} \\ &= G_1[k] - jG_2[k] \end{aligned}$$

where the second equality is due to g_1 and g_2 being real, and the third equality follows from $e^{j\left(\frac{2\pi}{N}\right)Nn} = 1$.

2. We know that

$$\begin{aligned} X[k] &= G_1[k] + jG_2[k], \\ X^*[-k \mod N] &= G_1[k] - jG_2[k]. \end{aligned}$$

Therefore,

$$\begin{aligned} G_1[k] &= \frac{1}{2}\left(X[k] + X^*[-k \mod N]\right), \\ G_2[k] &= \frac{1}{2j}\left(X[k] - X^*[-k \mod N]\right). \end{aligned}$$

3.

$$\begin{aligned} V[k] &= \sum_{n=0}^{2N-1} v[n] e^{-j\left(\frac{2\pi}{2N}\right)kn} \\ &= \sum_{n=0}^{N-1} v[2n] e^{-j\left(\frac{2\pi}{2N}\right)k2n} + \sum_{n=0}^{N-1} v[2n+1] e^{-j\left(\frac{2\pi}{2N}\right)k(2n+1)} \\ &= \sum_{n=0}^{N-1} g_1[n] e^{-j\left(\frac{2\pi}{N}\right)kn} + \sum_{n=0}^{N-1} g_2[n] e^{-j\left(\frac{2\pi}{N}\right)kn} e^{-j\left(\frac{2\pi}{2N}\right)k} \\ &= G_1[k] + e^{-j\left(\frac{\pi}{N}\right)k} G_2[k] \end{aligned}$$

Problem 2.

1.

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\left(\frac{2\pi}{N}\right)kn}$$

$$\begin{aligned}
\tilde{X}_3[k] &= \sum_{n=0}^{3N-1} \tilde{x}[n] e^{-j\left(\frac{2\pi}{3N}\right)kn} \\
&= \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\left(\frac{2\pi}{3N}\right)kn} + \sum_{n=N}^{2N-1} \tilde{x}[n] e^{-j\left(\frac{2\pi}{3N}\right)kn} + \sum_{n=2N}^{3N-1} \tilde{x}[n] e^{-j\left(\frac{2\pi}{3N}\right)kn} \\
&= \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\left(\frac{2\pi}{3N}\right)kn} + \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\left(\frac{2\pi}{3N}\right)k(n+N)} \\
&\quad + \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\left(\frac{2\pi}{3N}\right)k(n+2N)} \\
&= \left(1 + e^{-j\left(\frac{2\pi}{3}\right)k} + e^{-j\left(\frac{2\pi}{3}\right)2k}\right) \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\left(\frac{2\pi}{3N}\right)kn} \\
&= \left(1 + 2(-1)^k \cos\left(\frac{\pi k}{3}\right)\right) \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\left(\frac{2\pi}{N}\right)\frac{k}{3}n} \\
&= \begin{cases} 3\tilde{X}[k/3], & \text{for } k = 3l \\ 0, & \text{else} \end{cases}
\end{aligned}$$

2. The signal in question signal has period 2, therefore with $N = 2$ the DFT is

$$\begin{aligned}
\tilde{X}[k] &= \sum_{n=0}^{1} \tilde{x}[n] e^{-j\left(\frac{2\pi}{2}\right)kn} \\
&= (1 + 2e^{-j\left(\frac{2\pi}{2}\right)k}) \\
&= (1 + 2(-1)^k) \\
&= \begin{cases} 3, & \text{for } k = 0 \\ -1, & \text{for } k = 1 \end{cases}
\end{aligned}$$

For $N = 6$ the DFT becomes,

$$\begin{aligned}
\tilde{X}_3[k] &= \sum_{n=0}^{5} \tilde{x}[n] e^{-j\left(\frac{2\pi}{6}\right)kn} \\
&= \left(1 + e^{-j\left(\frac{2\pi}{6}\right)2k} + e^{-j\left(\frac{2\pi}{6}\right)4k}\right) + 2\left(e^{-j\left(\frac{2\pi}{6}\right)k} + e^{-j\left(\frac{2\pi}{6}\right)3k} + e^{-j\left(\frac{2\pi}{6}\right)5k}\right) \\
&= \left(1 + e^{-j\left(\frac{2\pi}{6}\right)2k} + e^{-j\left(\frac{2\pi}{6}\right)4k}\right) + 2e^{-j\left(\frac{2\pi}{6}\right)k}\left(1 + e^{-j\left(\frac{2\pi}{6}\right)2k} + e^{-j\left(\frac{2\pi}{6}\right)4k}\right) \\
&= \left(1 + e^{-j\left(\frac{2\pi}{3}\right)k} + e^{-j\left(\frac{2\pi}{3}\right)2k}\right)(1 + 2(-1)^{\frac{k}{3}}) \\
&= \begin{cases} 9, & \text{for } k = 0 \\ -3, & \text{for } k = 3 \\ 0, & \text{for } k = 1, 2, 4, 5 \end{cases}
\end{aligned}$$

Problem 3.

1.

$$\tilde{X}_1[k] = \sum_{n=0}^{14} e^{-3n} e^{-j\frac{2\pi}{15}kn} = \sum_{n=0}^{14} \left(e^{-3n} e^{-j\frac{2\pi}{15}k}\right)^n = \frac{1-e^{-45}}{1-e^{-(3+j\frac{2\pi}{15}k)}}$$

2.

$$\begin{aligned}
\tilde{X}_2[k] &= \sum_{n=0}^{1} \tilde{x}_2[n] e^{-j\left(\frac{2\pi}{2}\right)kn} \\
&= 1 - e^{-j\left(\frac{2\pi}{2}\right)k} \\
&= 1 - (-1)^k \\
&= \begin{cases} 0, & \text{for } k \text{ even} \\ 2, & \text{for } k \text{ odd} \end{cases}
\end{aligned}$$

3.

$$\begin{aligned}
\tilde{X}_3[k] &= \sum_{n=0}^{3} \tilde{x}_3[n] e^{-j\left(\frac{2\pi}{4}\right)kn} \\
&= (1 + e^{-j\left(\frac{2\pi}{4}\right)k}) - (e^{-j\left(\frac{2\pi}{4}\right)2k} + e^{-j\left(\frac{2\pi}{4}\right)3k}) \\
&= (1-(-1)^k)(1+e^{-j\left(\frac{\pi}{2}\right)k}) \\
&= \begin{cases} 0, & \text{for } k \text{ even} \\ 2(1+(-j)^k), & \text{for } k \text{ odd} \end{cases}
\end{aligned}$$

Problem 4.

1.

$$\begin{aligned}
\left(\frac{1}{8}\sum_{k=0}^{7} X[k] e^{j\left(\frac{2\pi}{8}\right)kn}\right)\Bigg|_{n=9} &= \frac{1}{8}\sum_{k=0}^{7} X[k] e^{j\left(\frac{2\pi}{8}\right)k(8+1)} \\
&= \sum_{k=0}^{7} X[k] e^{j\left(\frac{2\pi}{8}\right)k} = x[1].
\end{aligned}$$

2.

$$\begin{aligned}
w[n] &= \frac{1}{4}\sum_{k=0}^{3} X[k] e^{j\left(\frac{2\pi}{4}\right)kn} + \frac{1}{4}\sum_{k=0}^{3} X[k+4] e^{j\left(\frac{2\pi}{4}\right)kn} \\
&= \frac{1}{4}\sum_{k=0}^{3} X[k] e^{j\left(\frac{2\pi}{4}\right)kn} + \frac{1}{4}\sum_{k=4}^{7} X[k] e^{j\left(\frac{2\pi}{4}\right)(k-4)n} \\
&= \frac{1}{4}\sum_{k=0}^{3} X[k] e^{j\left(\frac{2\pi}{4}\right)kn} + \frac{1}{4}\sum_{k=4}^{7} X[k] e^{j\left(\frac{2\pi}{4}\right)kn} \quad (e^{j\left(\frac{2\pi}{4}\right)4n} = 1) \\
&= \frac{1}{4}\sum_{k=0}^{7} X[k] e^{j\left(\frac{2\pi}{8}\right)k(2n)} \\
&= 2x[2n].
\end{aligned}$$

3. Note that $Y[k] = X[k] + (-1)^k X[k]$. Therefore we have

$$\begin{aligned}
y[n] &= \frac{1}{8}\sum_{k=0}^{7} X[k]e^{j\left(\frac{2\pi}{8}\right)kn} + \frac{1}{8}\sum_{k=0}^{7} X[k](-1)^k e^{j\left(\frac{2\pi}{8}\right)kn} \\
&= \frac{1}{8}\sum_{k=0}^{7} X[k]e^{j\left(\frac{2\pi}{8}\right)kn} + \frac{1}{8}\sum_{k=0}^{7} X[k]e^{j\left(\frac{2\pi}{8}\right)k(n+4)} \\
&= x[n] + x[n+4 \mod 8].
\end{aligned}$$