JOINT SINGULAR VALUE DISTRIBUTION OF TWO CORRELATED RECTANGULAR GAUSSIAN MATRICES AND ITS APPLICATION

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Abstract. Let $\mathbf{H} = (h_{ij})$ and $\mathbf{G} = (g_{ij})$ be two $m \times n$, $m \leq n$, random matrices, each with i.i.d complex zero-mean unit-variance Gaussian entries, with correlation between any two elements given by $\mathbb{E}[h_{ij} g_{pq}^{\star}] = \rho\, \delta_{ip} \delta_{jq}$ such that $|\rho| < 1$, where * denotes the complex conjugate and δ_{ij} is the Kronecker delta. Assume $\{s_k\}_{k=1}^{m}$ and $\{r_l\}_{l=1}^{m}$ are unordered singular values of $\mathbf{H}$ and $\mathbf{G}$, respectively, and s and r are randomly selected from $\{s_k\}_{k=1}^{m}$ and $\{r_l\}_{l=1}^{m}$, respectively. In this paper, exact analytical closed-form expressions are derived for the joint probability distribution function (PDF) of $\{s_k\}_{k=1}^{m}$ and $\{r_l\}_{l=1}^{m}$ using an Itzykson-Zuber-type integral, as well as the joint marginal PDF of s and r, by a bi-orthogonal polynomial technique. These PDFs are of interest in multiple-input multiple-output (MIMO) wireless communication channels and systems.

Key words. correlated complex random matrices, joint singular value distribution, bi-orthogonal polynomials

AMS subject classifications. 15A52, 15A18, 62E15, 33C45

1. Introduction. Random singular values have found numerous applications such as hypothesis testing and principal component analysis in statistics [13], nuclear energy levels and level spacing in nuclear physics [11], and calculation of the multiple-input multiple-output (MIMO) channel capacity in wireless communications [17]. The singular value distribution of a *single* Gaussian random matrix is given in [15]. However, the joint singular value distribution of *correlated* Gaussian random matrices have received less attention so far, although it has important applications in wireless MIMO communications, say, the second-order statistics of the *eigen*-channels [22] and instantaneous mutual information [20, 21, 23].

To the best of our knowledge, correlated random matrices have been studied to some extent [4, 11, 12], where only Hermitian matrices were considered. Different from [4, 11, 12], we consider the situation where the elements, with the same indices, of the two rectangular complex Gaussian random matrices are correlated by a *complex* number, and derive exact analytical closed-form expressions for the joint PDF of their singular values.

This paper is organized as follows. Section 2 introduces the two rectangular complex Gaussian random matrices. The joint PDF of singular values are studied in section 3 using an Itzykson-Zuber-type integral. The joint marginal PDF of singular values is derived in section 4 and its application to wireless MIMO communications is presented in section 5. Finally, concluding remarks are summarized in section 6.

Notation: $\cdot^{\dagger}$ is reserved for matrix Hermitian, $\cdot^{T}$ for matrix transpose, $\cdot^{\star}$ for complex conjugate, $\mathrm{tr}[\cdot]$ for the trace of a matrix, $\jmath$ for $\sqrt{-1}$, $\mathbb{E}[\cdot]$ for mathematical expectation, $\mathbf{I}_m$ for the $m \times m$ identity matrix, $\otimes$ for the Kronecker product, and $\Re[\cdot]$ and $\Im[\cdot]$ for the real and imaginary parts of a complex number, respectively. In addition, $\mathrm{diag}(\mathbf{s})$ denotes a diagonal matrix with $\mathbf{s}$ on the main diagonal, $t \in [m, n]$ implies that t, m, and n are all integers such that $m \leq t \leq n$ with $m \leq n$, and $\det |x_{kl}|$ is the determinant of the matrix, where x_{kl} resides on the k^{th} row and l^{th} column. Moreover, lower-case bold letters represent row vectors, whereas upper-case

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bold letters are used for matrices. Finally $\mathcal{CN}$ means complex normal, and $\mathrm{vec}(\cdot)$ stacks all the columns of its matrix argument into one tall column vector.

2. Problem Description. There are two $m \times n$ random matrices $\mathbf{H} = (h_{ij})$ and $\mathbf{G} = (g_{ij})$, $i \in [1, m]$, $j \in [1, n]$, each with i.i.d complex zero-mean unit-variance Gaussian entries, i.e., $\mathbb{E}[h_{ij}] = \mathbb{E}[g_{ij}] = 0, \forall i, j$, $\mathbb{E}[h_{ij} h^\star_{pq}] = \mathbb{E}[g_{ij} g^\star_{pq}] = \delta_{ip}\delta_{jq}$, where the Kronecker symbol δ_{ij} is 1 or 0 when $i = j$ or $i \neq j$. Therefore $\mathbf{H}, \mathbf{G} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{mn})$. Moreover, the correlation among the two random matrices is given by

$$\mathbb{E}[h_{ij} g^\star_{pq}] = \rho\, \delta_{ip}\delta_{jq}, \quad \forall i, j, p, q, \tag{2.1}$$

where $\rho = |\rho| e^{\jmath\theta}$ is a complex number with $|\rho| < 1$.

Without loss of generality, we assume $m \leq n$ and set $\nu = n - m$. Based on the singular value decomposition (SVD), $\mathbf{H}$ and $\mathbf{G}$ can be, respectively, diagonalized as [7]

$$\mathbf{H} = \mathbf{U}\mathbf{S}\mathbf{V}^\dagger, \tag{2.2}$$

$$\mathbf{G} = \widetilde{\mathbf{U}}\mathbf{R}\widetilde{\mathbf{V}}^\dagger, \tag{2.3}$$

where $\mathbf{S} = \big[\mathrm{diag}(\mathbf{s})\ \mathbf{0}\big]$ and $\mathbf{R} = \big[\mathrm{diag}(\mathbf{r})\ \mathbf{0}\big]$ with $\mathbf{s} = [s_1, s_2, \cdots, s_m]$ and $\mathbf{r} = [r_1, r_2, \cdots, r_m]$, respectively.

We assume that the singular values of $\mathbf{G}$, $r_1, r_2, \cdots, r_m$, are unordered and the singular values of $\mathbf{H}$, $s_1, s_2, \cdots, s_m$, are also unordered. Now we would like to know the joint PDF of $\{r_l\}_{l=1}^m$ and $\{s_l\}_{l=1}^m$. Moreover, with r randomly selected from $r_1, r_2, \cdots, r_m$, and s randomly selected from $s_1, s_2, \cdots, s_m$, it is of interest to derive the joint PDF of r and s as well. These two PDFs are derived in Section 3 and 4, respectively.

3. Joint PDF of $\{s_l\}_{l=1}^m$ and $\{r_l\}_{l=1}^m$. Lemma 3.1 (Joint PDF of $\mathbf{H}$ and $\mathbf{G}$). *For two correlated rectangular complex Gaussian random matrices, $\mathbf{H}, \mathbf{G} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{mn})$, with the correlation between $\mathbf{H}$ and $\mathbf{G}$ given by (2.1), the joint PDF of $\mathbf{H}$ and $\mathbf{G}$ is given by*

$$p(\mathbf{H}, \mathbf{G}) = \frac{1}{\pi^{2mn}\left(1 - |\rho|^2\right)^{mn}} \exp\left[-\frac{\mathrm{tr}\big(\mathbf{H}\mathbf{H}^\dagger + \mathbf{G}\mathbf{G}^\dagger - \rho^\star \mathbf{H}\mathbf{G}^\dagger - \rho\mathbf{G}\mathbf{H}^\dagger\big)}{1 - |\rho|^2}\right]. \tag{3.1}$$

Proof. We set $\mathbf{h} = \mathrm{vec}(\mathbf{H})$, $\mathbf{g} = \mathrm{vec}(\mathbf{G})$, and $\mathbf{x} = [\mathbf{h}^T\ \mathbf{g}^T]^T$. Based on $\mathbf{H}, \mathbf{G} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{mn})$ and (2.1), we have the mean and covariance matrix of $\mathbf{x}$ as $\mathbb{E}[\mathbf{x}] = \mathbf{0}$ and $\Sigma_{\mathbf{x}} = \Sigma_\tau \otimes \mathbf{I}_{mn}$ with $\Sigma_\tau = \left[\begin{smallmatrix} 1 & \rho \\ \rho^\star & 1 \end{smallmatrix}\right]$, respectively. Therefore the PDF of $\mathbf{x}$ is given by [10]

$$p(\mathbf{x}) = \frac{1}{\pi^{2mn} \det|\Sigma_{\mathbf{x}}|} \exp\left(-\mathbf{x}^\dagger \Sigma_{\mathbf{x}}^{-1} \mathbf{x}\right), \tag{3.2}$$

where $\det|\Sigma_{\mathbf{x}}| = (\det|\Sigma_\tau|)^{mn} = \left(1 - |\rho|^2\right)^{mn}$.

With $\Sigma_\tau^{-1} = \frac{1}{1-|\rho|^2}\left[\begin{smallmatrix} 1 & -\rho \\ -\rho^\star & 1 \end{smallmatrix}\right]$, we obtain $\Sigma_{\mathbf{x}}^{-1} = \Sigma_\tau^{-1} \otimes \mathbf{I}_{mn} = \frac{1}{1-|\rho|^2}\left[\begin{smallmatrix} \mathbf{I}_{mn} & -\rho\mathbf{I}_{mn} \\ -\rho^\star\mathbf{I}_{mn} & \mathbf{I}_{mn} \end{smallmatrix}\right]$. Therefore $\mathbf{x}^\dagger \Sigma_{\mathbf{x}}^{-1} \mathbf{x}$ in (3.2) can be rewritten as

$$\begin{aligned} \mathbf{x}^\dagger \Sigma_{\mathbf{x}}^{-1} \mathbf{x} &= \mathrm{tr}\left(\Sigma_{\mathbf{x}}^{-1} \mathbf{x}\mathbf{x}^\dagger\right) = \mathrm{tr}\left(\frac{1}{1 - |\rho|^2}\left[\begin{smallmatrix} \mathbf{I}_{mn} & -\rho\mathbf{I}_{mn} \\ -\rho^\star\mathbf{I}_{mn} & \mathbf{I}_{mn} \end{smallmatrix}\right]\left[\begin{smallmatrix} \mathbf{h}\mathbf{h}^\dagger & \mathbf{h}\mathbf{g}^\dagger \\ \mathbf{g}\mathbf{h}^\dagger & \mathbf{g}\mathbf{g}^\dagger \end{smallmatrix}\right]\right), \\ &= \frac{\mathrm{tr}\left(\mathbf{h}\mathbf{h}^\dagger + \mathbf{g}\mathbf{g}^\dagger - \rho^\star\mathbf{h}\mathbf{g}^\dagger - \rho\mathbf{g}\mathbf{h}^\dagger\right)}{1 - |\rho|^2} = \frac{\mathrm{tr}\big(\mathbf{H}\mathbf{H}^\dagger + \mathbf{G}\mathbf{G}^\dagger - \rho^\star\mathbf{H}\mathbf{G}^\dagger - \rho\mathbf{G}\mathbf{H}^\dagger\big)}{1 - |\rho|^2}, \end{aligned} \tag{3.3}$$

where $\mathrm{tr}\left(\mathbf{A}\mathbf{B}^{\dagger}\right) = \mathrm{vec}(\mathbf{B})^{\dagger}\,\mathrm{vec}(\mathbf{A}) = \mathrm{tr}\left[\mathrm{vec}(\mathbf{A})\,\mathrm{vec}(\mathbf{B})^{\dagger}\right]$ [6] is used in the last "=" of (3.3). Substitution of (3.3) into (3.2) leads to (3.1). ∎

From (2.2), we know that the unitary matrix pair $(\mathbf{U}, \mathbf{V})$ parameterizes the coset space $\mathcal{U}(m) \times \mathcal{U}(n)/\left[\mathcal{U}(1)\right]^{m}$, where $\mathcal{U}(p)$ is the unitary group of order p, and the integration measure, $d[\mathbf{H}] = \prod_{i=1}^{m}\prod_{j=1}^{n} d\left[\Re h_{ij}\right] d\left[\Im h_{ij}\right]$, can be represented by [8]

$$d[\mathbf{H}] = \Omega J(\mathbf{s}) d[\mathbf{s}] d\mu(\mathbf{U}, \mathbf{V}), \tag{3.4}$$

where $J(\mathbf{s}) = \triangle^2(\mathbf{s}^2)\prod_{k=1}^{m} s_k^{2\nu+1}$ with the m-dimensional Vandermonde determinant $\triangle(\mathbf{s}^2) = \det\left|s_k^{2(l-1)}\right| = \prod_{k>l}(s_k^2 - s_l^2)$ and $\triangle^2(\cdot) = \left[\triangle(\cdot)\right]^2$, $d[\mathbf{s}] = \prod_{l=1}^{m} ds_l$, $d\mu(\mathbf{U}, \mathbf{V})$ is the Haar measure of $\mathcal{U}(m)\times\mathcal{U}(n)/\left[\mathcal{U}(1)\right]^{m}$ [8], and the constant Ω is given by [8,14]

$$\Omega = \frac{2^m \pi^{mn}}{\prod_{j=1}^{m} j!(j+\nu-1)!} = \frac{2^m \pi^{mn}}{m!\prod_{j=0}^{m-1} j!(j+\nu)!}. \tag{3.5}$$

Similarly, we have

$$d[\mathbf{G}] = \Omega J(\mathbf{r}) d[\mathbf{r}] d\mu(\widetilde{\mathbf{U}}, \widetilde{\mathbf{V}}), \tag{3.6}$$

where $J(\mathbf{r}) = \triangle^2(\mathbf{r}^2)\prod_{k=1}^{m} r_k^{2\nu+1}$ with the m-dimensional Vandermonde determinant $\triangle(\mathbf{r}^2) = \det\left|r_k^{2(l-1)}\right| = \prod_{k>l}(r_k^2 - r_l^2)$ and $d[\mathbf{r}] = \prod_{l=1}^{m} dr_l$.

In order to obtain the joint PDF of $\{r_l\}_{l=1}^{m}$ and $\{s_l\}_{l=1}^{m}$, we need the following proposition.

PROPOSITION 3.2 (An Itzykson-Zuber-type integral [8, (31)]).

$$\begin{aligned}\int d\mu(\mathbf{U}, \mathbf{V}) \exp\left\{-\frac{\mathrm{tr}\left[(\mathbf{H}-\mathbf{G})(\mathbf{H}-\mathbf{G})^{\dagger}\right]}{t}\right\} \\ = \frac{2^m \pi^{mn} t^{mn-m} \det\left|\exp\left(-\frac{s_k^2+r_l^2}{t}\right) I_\nu\left(\frac{2 s_k r_l}{t}\right)\right|}{m!\Omega\triangle(\mathbf{s}^2)\triangle(\mathbf{r}^2)\prod_{k=1}^{m}(s_k r_k)^{\nu}},\end{aligned} \tag{3.7}$$

where Ω is given by (3.5) and $I_k(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta}\cos(k\theta)d\theta$ is the k^{th} order modified Bessel function of the first kind.

THEOREM 3.3. *The joint PDF of the singular values of $\mathbf{H}$ and $\mathbf{G}$ is given by*

$$p(\mathbf{s}, \mathbf{r}) = \frac{\exp\left(-\frac{\sum_{k=1}^{m} s_k^2 + r_k^2}{1-|\rho|^2}\right)\triangle(\mathbf{s}^2)\triangle(\mathbf{r}^2)\prod_{k=1}^{m}(s_k r_k)^{\nu+1}\det\left|I_\nu\left(\frac{2|\rho| s_k r_l}{1-|\rho|^2}\right)\right|}{2^{-2m} m! m! \prod_{j=0}^{m-1} j!(j+\nu)! |\rho|^{mn-m}(1-|\rho|^2)^m}. \tag{3.8}$$

Proof. By combining (3.1) with (3.4) and (3.6), we obtain

$$p(\mathbf{s}, \mathbf{r}) = \frac{\Omega^2 J(\mathbf{s}) J(\mathbf{r})}{\pi^{2mn}(1-|\rho|^2)^{mn}}\Phi(\mathbf{s}, \mathbf{r}), \tag{3.9}$$

where

$$\begin{aligned}
\Phi(\mathbf{s},\mathbf{r}) &= \int d\mu(\widetilde{\mathbf{U}},\widetilde{\mathbf{V}}) \int d\mu(\mathbf{U},\mathbf{V}) \exp\left[-\frac{\mathrm{tr}\big(\mathbf{H}\mathbf{H}^\dagger + \mathbf{G}\mathbf{G}^\dagger - \rho^\star \mathbf{H}\mathbf{G}^\dagger - \rho\mathbf{G}\mathbf{H}^\dagger\big)}{1-|\rho|^2}\right], \\
&= \int d\mu(\widetilde{\mathbf{U}},\widetilde{\mathbf{V}}) \int d\mu(\mathbf{U},\mathbf{V}) \exp\left\{-\frac{\mathrm{tr}\left[(\mathbf{H}-\rho\mathbf{G})(\mathbf{H}-\rho\mathbf{G})^\dagger\right]}{1-|\rho|^2} - \mathrm{tr}(\mathbf{G}\mathbf{G}^\dagger)\right\}, \\
&= \int d\mu(\widetilde{\mathbf{U}},\widetilde{\mathbf{V}})\, e^{-\mathrm{tr}(\mathbf{G}\mathbf{G}^\dagger)} \int d\mu(\mathbf{U},\mathbf{V}) \exp\left\{-\frac{\mathrm{tr}\left[(\mathbf{H}-\rho\mathbf{G})(\mathbf{H}-\rho\mathbf{G})^\dagger\right]}{1-|\rho|^2}\right\}, \\
&= \int d\mu(\widetilde{\mathbf{U}},\widetilde{\mathbf{V}}) \frac{e^{-\sum_{k=1}^m r_k^2}(1-|\rho|^2)^{mn-m} \det\left| e^{-\frac{s_k^2+|\rho|^2 r_l^2}{1-|\rho|^2}} I_\nu\!\left(\frac{2|\rho| s_k r_l}{1-|\rho|^2}\right)\right|}{2^{-m} m! \pi^{-mn} \Omega \triangle(\mathbf{s}^2)\triangle(|\rho|^2\mathbf{r}^2)\prod_{k=1}^m (|\rho| s_k r_k)^\nu}, \\
&= \frac{(1-|\rho|^2)^{mn-m} \exp\left(-\frac{\sum_{k=1}^m s_k^2+r_k^2}{1-|\rho|^2}\right) \det\left| I_\nu\!\left(\frac{2|\rho| s_k r_l}{1-|\rho|^2}\right)\right|}{2^{-m} m! \pi^{-mn}\Omega |\rho|^{mn-m} \triangle(\mathbf{s}^2)\triangle(\mathbf{r}^2)\prod_{k=1}^m (s_k r_k)^\nu}.
\end{aligned} \tag{3.10}$$

Derivation of the second and third lines of (3.10) are straightforward. The fourth line comes from

$$\rho\mathbf{G} = \widehat{\mathbf{U}}\widehat{\mathbf{R}}\widehat{\mathbf{V}}^\dagger \tag{3.11}$$

with $\widehat{\mathbf{R}} = |\rho|\mathbf{R}$, and Proposition 3.2 with the replacements $t \to 1-|\rho|^2$ and $\mathbf{G} \to \rho\mathbf{G}$. The last line is based on the convention that $\int d\mu(\widetilde{\mathbf{U}},\widetilde{\mathbf{V}}) = 1$ [8]. Plugging (3.5) and the last line of (3.10) into (3.9), we obtain (3.8). □

By relating the eigenvalues of $\mathbf{G}\mathbf{G}^\dagger$ to the singular values of $\mathbf{G}$ through $\alpha_l = r_l^2$, $l \in [1, m]$, and the eigenvalues of $\mathbf{H}\mathbf{H}^\dagger$ to the singular values of $\mathbf{H}$ through $\beta_l = s_l^2$, $l \in [1, m]$, we can derive the joint PDF of $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \cdots, \alpha_m]$ and $\boldsymbol{\beta} = [\beta_1, \beta_2, \cdots, \beta_m]$, presented in the following corollary.

Corollary 3.4. *The joint PDF of the unordered eigenvalues of* $\mathbf{H}\mathbf{H}^\dagger$ *and* $\mathbf{G}\mathbf{G}^\dagger$ *is*

$$p(\boldsymbol{\beta},\boldsymbol{\alpha}) = \frac{\exp\left(-\frac{\sum_{k=1}^m \beta_k+\alpha_k}{1-|\rho|^2}\right)\triangle(\boldsymbol{\beta})\triangle(\boldsymbol{\alpha})\prod_{k=1}^m(\sqrt{\beta_k\alpha_k})^\nu \det\left| I_\nu\!\left(\frac{2|\rho|\sqrt{\beta_k\alpha_l}}{1-|\rho|^2}\right)\right|}{m!m!\prod_{j=0}^{m-1} j!(j+\nu)!|\rho|^{mn-m}(1-|\rho|^2)^m}, \tag{3.12}$$

where m-dimensional Vandermonde determinants are defined by $\triangle(\boldsymbol{\beta}) = \det\left|\beta_k^{l-1}\right| = \prod_{k>l}(\beta_k-\beta_l)$ *and* $\triangle(\boldsymbol{\alpha}) = \det\left|\alpha_k^{l-1}\right| = \prod_{k>l}(\alpha_k-\alpha_l)$.

Proof. It is straightforward to obtain (3.12) from (3.8) by $2m$ one-to-one nonlinear mappings. □

4. Joint Marginal PDF. In this section, with $\beta = s^2$ and $\alpha = r^2$, we calculate the joint marginal PDF of β and α, $p(\beta,\alpha)$, using the techniques and results presented in [4, 12]. Then the joint PDF of s and r, $p(s,r)$, is easily derived.

If the polynomials $P_k(\beta)$ and $Q_l(\alpha)$, satisfy $\int w(\beta,\alpha)P_k(\beta)Q_l(\alpha)d\beta d\alpha = \delta_{kl}$, then we call $P_k(\beta)$ and $Q_l(\alpha)$ as bi-orthogonal polynomials, associated with the weight function $w(\beta,\alpha)$ [11]. With this definition, we have the following Lemma.

LEMMA 4.1. *There exist bi-polynomials, $P_k(\beta)$ and $Q_l(\alpha)$, and a weight function, $w(\beta,\alpha)$, which reduce (3.12) to the following form*

$$p(\boldsymbol{\beta},\boldsymbol{\alpha}) = C_1 \det|P_{k-1}(\beta_l)| \det|w(\beta_k,\alpha_l)| \det|Q_{k-1}(\alpha_l)|, \tag{4.1}$$

where C_1 is a normalization constant.

Proof. In this paper, ν is a non-negative integer. Using the Hille-Hardy formula [2, pp. 185, (46)]

$$\sum_{k=0}^{\infty} \frac{k!z^k}{(k+\nu)!} L_k^\nu(x) L_k^\nu(y) = \frac{(xyz)^{-\frac{\nu}{2}}}{1-z} \exp\left(-z\frac{x+y}{1-z}\right) I_\nu\left(\frac{2\sqrt{xyz}}{1-z}\right), |z|<1, \tag{4.2}$$

with $L_k^\nu(x) = \frac{1}{k!} e^x x^{-\nu} \frac{d^k}{dx^k}(e^{-x}x^{k+\nu})$ as the associated Laguerre polynomial, we can rewrite (3.12) as

$$p(\boldsymbol{\beta},\boldsymbol{\alpha}) = \frac{\triangle(\boldsymbol{\beta})\triangle(\boldsymbol{\alpha}) \det\left|\beta_k^\nu e^{-\beta_k} \alpha_l^\nu e^{-\alpha_l} \sum_{j=0}^{\infty} \frac{j!|\rho|^{2j} L_j^\nu(\beta_k) L_j^\nu(\alpha_l)}{(j+\nu)!}\right|}{m!m! \prod_{j=0}^{m-1} j!(j+\nu)!|\rho|^{m(m-1)}}. \tag{4.3}$$

We set the weight function, $w(\beta,\alpha)$, as

$$\begin{aligned} w(\beta,\alpha) &= \beta^\nu \alpha^\nu e^{-(\beta+\alpha)} \sum_{j=0}^{\infty} \frac{j!|\rho|^{2j} L_j^\nu(\beta) L_j^\nu(\alpha)}{(j+\nu)!}, \\ &= \frac{(\beta\alpha)^{\frac{\nu}{2}} e^{-\frac{\beta+\alpha}{1-|\rho|^2}} I_\nu\left(\frac{2|\rho|\sqrt{\beta\alpha}}{1-|\rho|^2}\right)}{(1-|\rho|^2)|\rho|^\nu}. \end{aligned} \tag{4.4}$$

It is easy to check that the corresponding bi-orthogonal polynomials are given by

$$P_k(\beta) = \sqrt{\frac{k!}{(k+\nu)!}} |\rho|^{-k} L_k^\nu(\beta), \tag{4.5}$$

$$Q_l(\alpha) = \sqrt{\frac{l!}{(l+\nu)!}} |\rho|^{-l} L_l^\nu(\alpha), \tag{4.6}$$

using the following integral equality [2, pp. 267, 7.414.3]

$$\int_0^\infty e^{-x} x^\nu L_k^\nu(x) L_l^\nu(x) = \frac{(k+\nu)!}{k!} \delta_{kl} \tag{4.7}$$

Moreover, by the addition of multiples of rows of lower order which do not change the determinant of the Vandermonde matrix, then each of the rows can be expressed in terms of orthogonal polynomials with respect to the weight function $w(\beta,\alpha)$. Therefore two m-dimensional Vandermonde determinants, $\triangle(\boldsymbol{\beta})$ and $\triangle(\boldsymbol{\alpha})$, can be represented as

$$\triangle(\boldsymbol{\beta}) = \det\left|\beta_k^{l-1}\right| = C_2 \det\left|P_{k-1}(\beta_l)\right|, \tag{4.8}$$

$$\triangle(\boldsymbol{\alpha}) = \det\left|\alpha_k^{l-1}\right| = C_3 \det\left|Q_{k-1}(\alpha_l)\right|, \tag{4.9}$$

where we use the fact that the matrix transpose does not change the determinant, i.e., $\det|P_{l-1}(\beta_k)| = \det|P_{k-1}(\beta_l)|$ and $\det|Q_{l-1}(\beta_k)| = \det|Q_{k-1}(\beta_l)|$.

The coefficient of x^k in $L_k^\nu(x)$ is $\frac{(-1)^k}{k!}$, then the coefficient of x^k in $P_k(x)$ is $(-1)^k|\rho|^{-k}\frac{1}{\sqrt{k!(k+\nu)!}}$, therefore we have $C_2 = \prod_{j=0}^{m-1}(-1)^j|\rho|^j\sqrt{j!(j+\nu)!} = (-1)^{\frac{m(m-1)}{2}}$ $\times\sqrt{|\rho|^{m(m-1)}\prod_{j=0}^{m-1}j!(j+\nu)!}$, obtained by plugging (4.5) into (4.8). Similarly, substitution of (4.6) into (4.9) gives $C_3 = C_2$. Now the product of (4.8) and (4.9) results in

$$\triangle(\boldsymbol{\beta})\triangle(\boldsymbol{\alpha}) = |\rho|^{m(m-1)}\prod_{j=0}^{m-1}j!(j+\nu)!\det|P_{k-1}(\beta_l)|\det|Q_{k-1}(\alpha_l)|. \tag{4.10}$$

Based on (4.4) and (4.10), one can see that (4.3) is equal to (4.1) with $C_1 = \frac{1}{m!m!}$. ◻

THEOREM 4.2. *The joint PDF of β and α is given by*

$$\begin{aligned} p(\beta,\alpha) = &\frac{(\beta\alpha)^{\frac{\nu}{2}}e^{-\frac{\beta+\alpha}{1-|\rho|^2}}I_\nu\left(\frac{2|\rho|\sqrt{\beta\alpha}}{1-|\rho|^2}\right)}{m^2(1-|\rho|^2)|\rho|^\nu}\sum_{k=0}^{m-1}\frac{k!}{(k+\nu)!}\frac{L_k^\nu(\beta)L_k^\nu(\alpha)}{|\rho|^{2k}} \\ &+\frac{(\beta\alpha)^\nu e^{-(\beta+\alpha)}}{m^2}\sum_{0\le k<l}^{m-1}\left\{\frac{k!l!}{(k+\nu)!(l+\nu)!}\left\{\left[L_k^\nu(\beta)L_l^\nu(\alpha)\right]^2+\left[L_l^\nu(\beta)L_k^\nu(\alpha)\right]^2\right.\right. \\ &\left.\left.-\left[|\rho|^{2(l-k)}+|\rho|^{2(k-l)}\right]L_k^\nu(\beta)L_l^\nu(\beta)L_k^\nu(\alpha)L_l^\nu(\alpha)\right\}\right\}. \end{aligned} \tag{4.11}$$

Proof. Based on Lemma 4.1, and the results presented in [12, (3.7)] [4], $p(\beta,\alpha)$ can be expressed as

$$m^2\,p(\beta,\alpha) = w(\beta,\alpha)\sum_{k=0}^{m-1}P_k(\beta)Q_k(\alpha) + \sum_{0\le k<l}^{m-1}\det\begin{vmatrix}P_k(\beta) & \overline{P}_k(\alpha)\\ P_l(\beta) & \overline{P}_l(\alpha)\end{vmatrix}\det\begin{vmatrix}\overline{Q}_k(\beta) & Q_k(\alpha)\\ \overline{Q}_l(\beta) & Q_l(\alpha)\end{vmatrix}, \tag{4.12}$$

where $P_k(x)$ and $Q_k(x)$ are defined in (4.5) and (4.6), respectively, the weight function is presented in (4.4), and $\overline{P}_k(\alpha)$ and $\overline{Q}_l(\beta)$ are similarly defined as [12]

$$\overline{P}_k(\alpha) = \int P_k(\beta)w(\beta,\alpha)d\beta = \sqrt{\frac{k!}{(k+\nu)!}}\alpha^\nu e^{-\alpha}|\rho|^kL_k^\nu(\alpha), \tag{4.13}$$

$$\overline{Q}_l(\beta) = \int Q_l(\alpha)w(\beta,\alpha)d\alpha = \sqrt{\frac{l!}{(l+\nu)!}}\beta^\nu e^{-\beta}|\rho|^lL_l^\nu(\beta). \tag{4.14}$$

Plugging (4.4), (4.5), (4.6), (4.13) and (4.14) into (4.12), we arrive at (4.11). ◻

It is straightforward to obtain the joint PDF of s and r from (4.11), according to these one-to-one mappings $s=\sqrt{\beta}$ and $r=\sqrt{\alpha}$.

The joint PDF in (4.11) includes many existing PDF's as special cases.

- By integration over β, (4.11) reduces to the marginal PDF

$$p(\alpha) = \frac{1}{m}\sum_{k=0}^{m-1}\frac{k!}{(k+\nu)!}\left[L_k^\nu(\alpha)\right]^2\alpha^\nu e^{-\alpha}, \tag{4.15}$$

which is the same as the PDF presented in [17]. When $m=1$, (4.15) further reduces to

$$p(\alpha)=\frac{1}{(n-1)!}\alpha^{n-1}e^{-\alpha}, \tag{4.16}$$

which is the χ^2 distribution with $2n$ degrees of freedom [16, (2.32)].

- With $m=1$, (4.11) reduces to [19],

$$p(\alpha,\beta)=\frac{(\alpha\beta)^{\frac{n-1}{2}}\exp\left(-\frac{\alpha+\beta}{1-|\rho|^2}\right)I_{n-1}\left(\frac{2|\rho|\sqrt{\alpha\beta}}{1-|\rho|^2}\right)}{(n-1)!\,(1-|\rho|^2)\,|\rho|^{n-1}}. \tag{4.17}$$

Furthermore, when $n=1$, (4.17) simplifies to

$$p(\alpha,\beta)=\frac{1}{1-|\rho|^2}\exp\left(-\frac{\alpha+\beta}{1-|\rho|^2}\right)I_0\left(\frac{2|\rho|\sqrt{\alpha\beta}}{1-|\rho|^2}\right), \tag{4.18}$$

which is identical to (8-103) [3, pp. 163], after two one-to-one nonlinear transformations.

For the application discussed in section 5, we need the joint marginal PDF of ϕ and φ, $p(\phi,\varphi)$, where ϕ and φ are randomly selected from $\{\alpha_k\}_{k=1}^m$, $m\geq 2$. Using the technique in [4, 12], we have the following theorem.

THEOREM 4.3. *If ϕ and φ are randomly selected from $\{\alpha_k\}_{k=1}^m$, their joint PDF is given by*

$$p(\phi,\varphi)=\frac{(\phi\varphi)^{\nu}e^{-(\phi+\varphi)}}{m(m-1)}\sum_{\substack{k,l=0\\k\neq l}}^{m-1}\frac{k!l!}{(k+\nu)!(l+\nu)!}\Big\{\left[L_k^{\nu}(\phi)L_l^{\nu}(\varphi)\right]^2-L_k^{\nu}(\phi)L_l^{\nu}(\phi)L_k^{\nu}(\varphi)L_l^{\nu}(\varphi)\Big\}. \tag{4.19}$$

Proof. According to (1.6) and (2.14) in [12] we have

$$p(\phi,\varphi)=\frac{1}{m(m-1)}\det\begin{vmatrix}K(\phi,\phi) & K(\phi,\varphi)\\ K(\varphi,\phi) & K(\varphi,\varphi)\end{vmatrix}, \tag{4.20}$$

where $K(x_1,x_2)=\sum_{k=0}^{m-1}P_k(x_1)\overline{Q}_k(x_2)$. With $P_k(x_1)$ in (4.5) and $\overline{Q}_k(x_2)$ in (4.14), we obtain (4.19) after some simple algebraic manipulations. □

5. Application to Wireless MIMO Communication Systems. For an $N_R\times N_T$ MIMO time-varying Rayleigh flat fading channel [18] with N_T transmitters and N_R receivers, the channel impulse response at time instant t is given by

$$\mathbf{H}(t)=\begin{bmatrix}h_{1,1}(t) & \cdots & h_{1,N_T}(t)\\ \vdots & \ddots & \vdots\\ h_{N_R,1}(t) & \cdots & h_{N_R,N_T}(t)\end{bmatrix}. \tag{5.1}$$

We assume all the N_RN_T sub-channels in the MIMO system, $\{h_{i,j}(t)\}_{(i=1,j=1)}^{(N_R,N_T)}$ are i.i.d., with the same temporal correlation coefficient, i.e.,

$$\mathbb{E}[h_{ij}(t)h_{pq}^{\star}(t-\tau)]=\delta_{ip}\delta_{jq}\rho_h(\tau), \tag{5.2}$$

where $\rho_h(\tau) = J_0(2\pi f_D \tau)$ [9] in isotropic scattering environments[1], with $J_0(x) = I_0(-\jmath x)$ [5, pp. 961, 8.406.3] and f_D is the maximum Doppler frequency shift.

We set $n = \max(N_R, N_T)$ and $m = \min(N_R, N_T)$. According to (2.2), $\mathbf{H}(t)$ can be diagonalized as

$$\mathbf{H}(t) = \mathbf{U}(t)\mathbf{S}(t)\mathbf{V}^\dagger(t), \tag{5.3}$$

where $\mathbf{S}(t) = \begin{bmatrix}\mathrm{diag}(\mathbf{s}(t)) & \mathbf{0}\end{bmatrix}$ with $\mathbf{s}(t) = [s_1(t), s_2(t), \cdots, s_m(t)]$ for $N_R \le N_T$, and $\mathbf{S}(t) = \begin{bmatrix}\mathrm{diag}(\mathbf{s}(t)) \\ \mathbf{0}\end{bmatrix}$ for $N_R > N_T$. Therefore the MIMO channel, $\mathbf{H}(t)$, is decomposed to m identically distributed *eigen*-channels $\lambda_k(t) = s_k^2(t)$, $k \in [1, m]$, by SVD.

In wireless MIMO communication systems, we are interested in the correlation coefficient between any two *eigen*-channels, which is defined by

$$\rho_{k,l}(\tau) = \frac{\mathbb{E}\left[\lambda_k(t)\lambda_l(t-\tau)\right] - \mathbb{E}\left[\lambda_k(t)\right]\mathbb{E}\left[\lambda_l(t)\right]}{\sqrt{\mathbb{E}\left[\lambda_k^2(t)\right] - \left\{\mathbb{E}\left[\lambda_k(t)\right]\right\}^2}\sqrt{\mathbb{E}\left[\lambda_l^2(t)\right] - \left\{\mathbb{E}\left[\lambda_l(t)\right]\right\}^2}}. \tag{5.4}$$

For simplicity, in this paper we only consider a 2×2 MIMO system, $N_R = N_T = 2$, where the correlation coefficient, $\rho_{k,l}(\tau)$, can be shown to be

$$\rho_{k,l}(\tau) = \begin{cases} 1 - \frac{3}{2}\left(1 - \delta_{kl}\right), & \tau = 0, \\ \frac{|\rho_h(\tau)|^2}{4} = \frac{J_0^2(2\pi f_D \tau)}{4}, & \tau \ne 0, \end{cases}, k, l = 1, 2, \tag{5.5}$$

with $J_0^2(\cdot) = [J_0(\cdot)]^2$. To derive (5.5), we note that for $\tau = 0$ and $k = l$, $\rho_{k,l}(0) = 1$ because of the definition of the correlation coefficient. Since $m = 2$, for any *eigen*-channel at the time instant t, it is easy to show that the mean value of $\lambda_k(t)$ is $\mathbb{E}\left[\lambda_k(t)\right] = 2$, $\forall k$, and the second moment of $\lambda_k(t)$ is $\mathbb{E}\left[\lambda_k^2(t)\right] = 8$, $\forall k$, using the PDF in (4.15). For $\tau = 0$ and $k \ne l$, we obtain $\mathbb{E}\left[\lambda_k(t)\lambda_l(t)\right] = 2$ by (4.19), hence $\rho_{k,l}(0) = -\frac{1}{2}$, $\forall k \ne l$. For $\tau \ne 0$ and $\forall k, l$, it is not difficult to get $\mathbb{E}\left[\lambda_k(t)\lambda_l(t-\tau)\right] = 4 + |\rho_h(\tau)|^2$ using (4.11), therefore we have the second line in (5.5).

Monte Carlo simulations are performed to verify the result in (5.5). In all simulations[2], the maximum Doppler frequency f_D is set to 1 Hz, and the sampling period, T_s, is equal to $\frac{1}{1000 f_D}$. The simulation results are shown in Fig. 5.1, where the upper figure shows the channel correlation coefficient $\rho_h(\tau) = J_0\left(2\pi f_D \tau\right)$, Clarke's correlation model, whereas the lower figure presents the correlation coefficient between any two *eigen*-channels or for any individual eigen-channel, Eq. (5.5). Since $J_0(2\pi f_D \tau)$ is an even function of τ, the correlation coefficients are plotted for $\tau \ge 0$. In all figures, "Simu." indicates the curve is obtained by Monte Carlo simulations, whereas "Theo." means theoretical. From Fig. 5.1 we can conclude that the new theoretical result in (5.5) is confirmed by simulation very well.

6. Conclusion. In this paper, the joint distribution of singular values of two correlated rectangular complex Gaussian random matrices is derived, as well as the joint marginal distribution. The derived distributions play an important role in the analysis and design of wireless MIMO communication systems. As an example, the correlation coefficient of any two *eigen*-channels of a 2×2 MIMO system is obtained and verified by the Monte Carlo simulations in this paper.

[1]In the non-isotropic scattering environment, $\rho_h(\tau)$, in general, is a complex-value function [20, 21], and $|\rho_h(\tau)|$ indicates its amplitude at the time delay τ.

[2]The spectral method [1] is used to generate the MIMO channels.

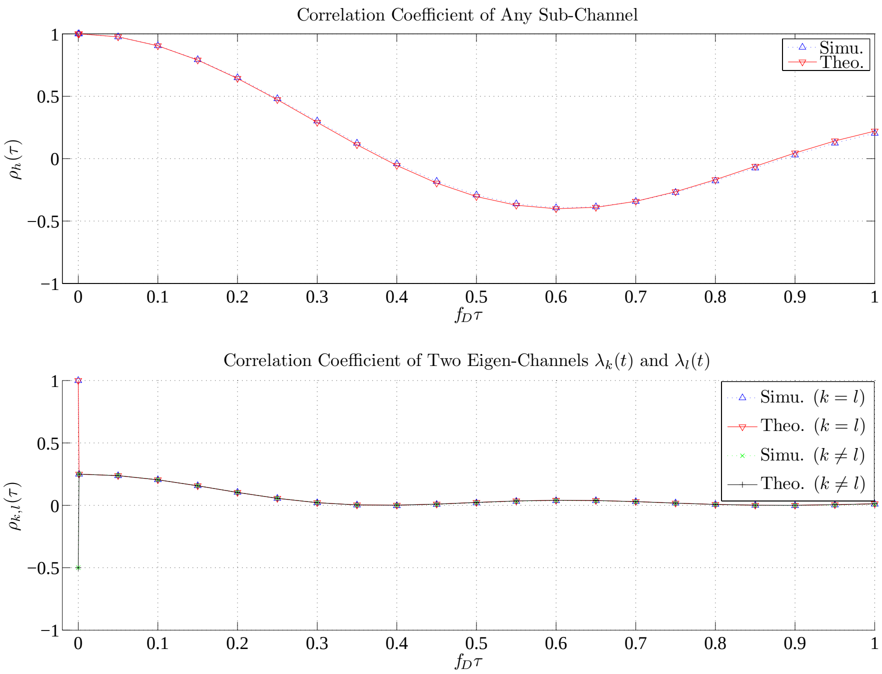

FIG. 5.1. *The channel correlation coefficient, $\rho_h(\tau)$, and correlation coefficient of any two eigen-channels, $\rho_{k,l}(\tau)$, in a 2×2 MIMO system with Clarke's correlation model. Note that the sampling period, T_s, is $\frac{1}{1000 f_D}$ in Monte Carlo simulations, therefore the first non-zero τ is T_s, i.e., $\frac{1}{1000 f_D}$, which corresponds to $f_D\tau = \frac{1}{1000}$ in the horizontal axis.*

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