Solutions 7

1.a) We have:

$$\operatorname{Re} g_F(u+iv) = \int_{\mathbb{R}} \frac{x-u}{(x-u)^2+v^2}\, dF(x) \quad \text{and} \quad \operatorname{Im} g_F(u+iv) = \int_{\mathbb{R}} \frac{v}{(x-u)^2+v^2}\, dF(x).$$

b) No proof required: the analyticity of g_F on $\mathbb{C}\backslash\mathbb{R}$ follows from the analyticity of $z \mapsto \frac{1}{x-z}$ on $\mathbb{C}\backslash\mathbb{R}$ and the use of the dominated convergence theorem.

c) If $v > 0$, then $\operatorname{Im} g_F(u+iv)$ is clearly positive by the above formula.

d) We have:

$$v^2\, |g_F(iv)|^2 = \left(\int_{\mathbb{R}} \frac{v(x-u)}{(x-u)^2+v^2}\, dF(x)\right)^2 + \left(\int_{\mathbb{R}} \frac{v^2}{(x-u)^2+v^2}\, dF(x)\right)^2.$$

By the dominated convergence theorem, the first term on the right-hand side converges to 0 as $v \to +\infty$ and the second term converges to 1.

e) This is a straightforward computation.

2.a) We have

$$\begin{aligned}
\frac{1}{\pi} \lim_{\varepsilon\downarrow 0} \int_a^b \operatorname{Im} g_F(x+i\varepsilon)\, dx &= \frac{1}{\pi} \lim_{\varepsilon\downarrow 0} \int_a^b \left(\int_{\mathbb{R}} \frac{\varepsilon}{(y-x)^2+\varepsilon^2}\, dF(y)\right) dx \\
&= \frac{1}{\pi} \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}} \left(\int_a^b \frac{\varepsilon}{(y-x)^2+\varepsilon^2}\, dx\right) dF(y) \\
&= \frac{1}{\pi} \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}} \operatorname{arctg}\left(\frac{y-x}{\varepsilon}\right)\Bigg|_{x=a}^{x=b} dF(y).
\end{aligned}$$

Since

$$\lim_{\varepsilon\downarrow 0} \operatorname{arctg}\left(\frac{y-x}{\varepsilon}\right)\Bigg|_{x=a}^{x=b} = \begin{cases} \pi, & \text{if } a<y<b, \\ \frac{\pi}{2}, & \text{if } y=a \text{ or } b \\ 0, & \text{otherwise} \end{cases} = \pi\,(1_{]a,b[}(y) + \frac{1}{2}\, 1_{\{a,b\}}(y)),$$

we conclude by the dominated convergence theorem that

$$\frac{1}{\pi} \lim_{\varepsilon\downarrow 0} \int_a^b \operatorname{Im} g_F(x+i\varepsilon)\, dx = F(b) - F(a),$$

at any $a<b$ continuity points of F.

b) Assuming that F has a pdf p_F, the very same computation as above leads to

$$\begin{aligned}
\frac{1}{\pi} \lim_{\varepsilon\downarrow 0} \int_a^b \operatorname{Im} g_F(x+i\varepsilon)\, dx &= \frac{1}{\pi} \lim_{\varepsilon\downarrow 0} \int_a^b \left(\int_{\mathbb{R}} \frac{\varepsilon}{(y-x)^2+\varepsilon^2}\, p_F(y)\, dy\right) dx \\
&= \int_{\mathbb{R}} (1_{]a,b[}(y) + \frac{1}{2}\, 1_{\{a,b\}}(y))\, p_F(y)\, dy = \int_a^b p_F(y)\, dy.
\end{aligned}$$

3.a) We have for $z = x + iy$,

$$g_0(z) = \frac{1}{x_0 - iy_0 - x - iy} = \frac{x_0 - x + i(y_0 + y)}{(x_0 - x)^2 + (y_0 + y)^2},$$

so

$$\mathrm{Im}(g_0(z)) = \frac{(y_0 + y)}{(x_0 - x)^2 + (y_0 + y)^2}.$$

Let us first consider the case where $y_0 = 0$. Then the result of Exercise 2, part a), tells us that for any $a < x_0 < b$,

$$\begin{aligned} F(]a,b[) &= \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_a^b \frac{\varepsilon}{(x_0 - x)^2 + \varepsilon^2}\, dx \\ &= \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \operatorname{arctg}\left(\frac{x - x_0}{\varepsilon}\right)\Bigg|_{x=a}^{x=b} = \frac{1}{\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 1. \end{aligned}$$

So that $F(b) - F(a) = 1$ for all $a < x_0 < b$, i.e. $F = 1_{\{x_0 \leq t\}}$ and the moments of F are $m_k = x_0^k$.

For the case $y_0 > 0$, we have

$$p_F(x) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \frac{y_0 + \varepsilon}{(x_0 - x)^2 + (y_0 + \varepsilon)^2} = \frac{1}{\pi} \frac{y_0}{(x_0 - x)^2 + y_0^2},$$

which is the Cauchy distribution with parameters x_0 and y_0. This distribution has no finite moments, but notice that x_0 is closely related to its "mean" and that $1/y_0$ is a measure of how spread the distribution is.

b) The solution of the equation is

$$g_\pm(z) = -\frac{z}{2} \pm \sqrt{\frac{z^2}{4} - 1}$$

and for $\mathrm{Im} z > 0$, only g_+ satsifies $\mathrm{Im} g_+(z) > 0$. Therefore, by Exercise 2, part b), we have

$$\begin{aligned} p_F(x) &= \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \mathrm{Im}(g_+(z)) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \left(-\frac{\varepsilon}{2} + \mathrm{Im}\left(\sqrt{\frac{(x + i\varepsilon)^2}{4} - 1}\right)\right) \\ &= \frac{1}{\pi} \mathrm{Im}\left(\sqrt{\frac{x^2}{4} - 1}\right) = \frac{1}{2\pi} \sqrt{4 - x^2}\, 1_{\{|x| \leq 2\}}. \end{aligned}$$

c) The solution of the equation is

$$g_\pm(z) = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{z}}$$

and for $\mathrm{Im} z > 0$, only g_+ satsifies $\mathrm{Im} g_+(z) > 0$. Therefore,

$$\begin{aligned} p_F(x) &= \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \mathrm{Im}(g_+(z)) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \mathrm{Im}\left(\sqrt{\frac{1}{4} - \frac{1}{x + i\varepsilon}}\right) \\ &= \frac{1}{\pi} \mathrm{Im}\left(\sqrt{\frac{1}{4} - \frac{1}{x}}\right) = \frac{1}{\pi} \sqrt{\frac{1}{x} - \frac{1}{4}}\, 1_{\{0 < x \leq 4\}}. \end{aligned}$$

4. We have

$$((H - zI_n)^{-1})_{11} = \frac{\det((H - zI_n)(1,1))}{\det(H - zI_n)} = \frac{\det((H - zI_n)(1,1))}{\sum_{j=1}^{n}(-1)^{1+j}\,(h_{1j} - z\delta_{1j})\,\det((H - zI_n)(1,j))}.$$

The denominator is equal to

$$\begin{aligned}
&(m_{11} - z)\,\det((H - zI_n)(1,1)) + \sum_{j=2}^{n}(-1)^{1+j}\,h_{1j}\,\det((H - zI_n)(1,j)) \\
&= (h_{11} - z)\,\det((H - zI_n)(1,1)) + \sum_{j,k=2}^{n}(-1)^{1+j+k}\,h_{1j}\,h_{k1}\,\det((H - zI_n)(1k,1j)) \\
&= (h_{11} - z)\,\det((H - zI_n)(1,1)) - \sum_{j,k=2}^{n} h_{1j}\,((H - zI_n)(1,1)^{-1})_{jk}\,h_{k1}\,\det((H - zI_n)(1,1)).
\end{aligned}$$

Therefore,

$$((H - zI_n)^{-1})_{11} = \frac{1}{(h_{11} - z) - h_1^T\,(H_1 - zI_{n-1})^{-1}\,h_1}.$$