**Solutions 6**

**1. a)** Using Cauchy-Schwartz inequality, we obtain

$$|m_{2k+1}|^2 = \left| \int_{\mathbb{R}} dF(x)\, x^k\, x^{k+1} \right|^2 \leq \left( \int_{\mathbb{R}} dF(x)\, x^{2k} \right) \left( \int_{\mathbb{R}} dF(x)\, x^{2k+2} \right) = m_{2k}\, m_{2k+2},$$

so a condition on the growth of the even moments of $F$ ensures the same growth for the odd moments.

bf b) If $|m_k| \leq C^k$, then $m_{2k}^{1/2k} \leq C$, so

$$\limsup_{k\to\infty} \frac{1}{2k}\, (m_{2k})^{\frac{1}{2k}} = 0 < \infty.$$

If in turn the above lim sup is finite, then this means that there exists $C > 0$ such that

$$(m_{2k})^{\frac{1}{2k}} \leq C\, 2k, \quad \forall k \geq k_0 \text{ sufficiently large},$$

so

$$\sum_{k=0}^{\infty} m_{2k}^{-\frac{1}{2k}} \geq \sum_{k=k_0}^{\infty} \frac{1}{C\, 2k} = \infty.$$

**2. a)** First observe that the odd moments of $F$ vanish, since the distribution is symmetric (i.e. $p_F(-x) = p_F(x)$ for all $x \in \mathbb{R}$). By the indicated change of variable, we have for $m_{2k}$:

$$\begin{aligned} m_{2k} &= \int_{-2}^{2} x^{2k}\, \frac{1}{2\pi}\, \sqrt{4-x^2}\, dx = \frac{1}{\pi} \int_0^{\pi/2} (2\sin(t))^{2k}\, \sqrt{4 - 4\sin(t)^2}\, 2\cos(t)\, dt \\ &= \frac{2^{2k+2}}{\pi} \int_0^{\pi/2} \sin(t)^{2k}\, \cos(t)^2\, dt = \frac{2^{2k+2}}{\pi} \left( \int_0^{\pi/2} \sin(t)^{2k}\, dt - \int_0^{\pi/2} \sin(t)^{2(k+1)}\, dt \right). \end{aligned}$$

By integration by parts (with $u(t) = \sin^{2k-1}(t)$ and $v(t) = \sin(t)$), we have

$$a_{k+1} := \int_0^{\pi/2} \sin(t)^{2(k+1)}\, dt = (2k+1) \int_0^{\pi/2} \sin(t)^{2k}\, \cos^2(t)\, dt = (2k+1)\,(a_k - a_{k+1}),$$

so

$$a_{k+1} = \frac{2k+1}{2k+2}\, a_k = \ldots = \frac{(2k+1)\cdots 3\cdot 1}{(2k+2)\cdots 4\cdot 2}\, a_0 = \frac{(2k+1)!/(2^k\, k!)}{2^{k+1}\,(k+1)!}\, a_0 = \frac{(2k+1)!}{2^{2k+1}\, k!(k+1)!}\, a_0.$$

Since $a_0 = \frac{\pi}{2}$, we finally obtain

$$m_{2k} = \frac{2^{2k+2}}{\pi}\, \frac{a_{k+1}}{2k+1} = \frac{(2k)!}{k!(k+1)!} = \frac{1}{k+1} \begin{pmatrix} 2k \\ k \end{pmatrix},$$

Since

$$m_{2k} \leq \frac{(2k)!}{(k!)^2} \leq \frac{((2k)(2k-2)\cdots 2)^2}{(k!)^2} = \frac{(2^k\, k!)^2}{(k!)^2} \leq 4^k,$$

Carleman's condition is satisfied. An easier way to see this is to notice that $F$ has bounded support ($[-2,2]$).

b) The density of $\mathcal{N}(0,1)$ is $p_F(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$. Therefore $p_F'(x) = -x\, p_F(x)$ and we have by integration by parts:

$$\int_{\mathbb{R}} x\, f(x)\, p_F(x)\, dx = -\int_{\mathbb{R}} f(x)\, p_F'(x)\, dx = \int_{\mathbb{R}} f'(x)\, p_F(x)\, dx,$$

where we have used the fact that the boundary term vanishes to zero as $n \to \infty$ (since $f(x)$ is by assumption growing polynomially at infinity and $p_F(x), p_F'(x) \simeq \exp(-x^2/2)$) .

Notice that $m_1 = 0$ and $m_2 = 1$. We then have, by application of part the preceding formula,

$$m_{k+2} = \int_{\mathbb{R}} dx\, p_F(x)\, x\, x^{k+1} = (k+1) \int_{\mathbb{R}} dx\, p_F(x)\, x^{k+1} = (k+1)\, m_k.$$

From this, we deduce by induction that $m_{2k+1} = 0$ for all $k \geq 0$ (but this could have also been deduced from the fact the $F$ is symmetric) and that

$$m_{2k} = (2k-1)\cdots 3 \cdot 1 = \frac{(2k)!}{2^k\,(k!)}, \quad \text{for all } k \geq 0.$$

This moments satisfy Carleman's condition, since as before,

$$m_{2k} \leq \frac{((2k)(2k-2)\cdots 2)^2}{2^k\, k!} = 2^k\, k!$$

and $k! \leq k^{k+1/2}$ by Stirling's formula.

**c)** Using the change of variable $x = e^y$, we obtain

$$m_k = \int_0^\infty x^k\, p_F(x)\, dx = \int_{\mathbb{R}} e^{ky}\, p_F(e^y)\, e^y\, dy.$$

Noticing that $p_F(e^y)\, e^y = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right)$, we further obtain

$$m_k = \frac{1}{\sqrt{2\pi}}\, e^{k^2/2} \int_{\mathbb{R}} e^{-(y-k)^2/2}\, dy = e^{k^2/2}.$$

These moments do not satisfy Carleman's condition, since

$$\sum_{k=0}^{\infty} m_{2k}^{-\frac{1}{2k}} = \sum_{k=0}^{\infty} e^{-k} < \infty.$$

**d)** In order to satisfy $\lim_{t\to\infty} F(t) = 1$, we must have $C = 1/\sum_{j\in\mathbb{Z}} e^{-j^2/2}$. Let us compute the moments of $F$:

$$m_k = C \sum_{j\in\mathbb{Z}} e^{jk}\, e^{-j^2/2} = C\, e^{k^2/2} \sum_{j\in\mathbb{Z}} e^{-(j-k)^2/2} = e^{k^2/2}.$$

Notice that the moments of this distribution and the preceding are the same, even though the distributions are different.

**e)** By the change of variable $y = x^\lambda$, we have

$$\int_{\mathbb{R}} x^k\, dF(x) = c_\lambda \int_0^\infty x^k\, \exp(-x^\lambda)\, dx = c_\lambda\, \lambda \int_0^\infty y^{k/\lambda}\, e^{-y}\, y^{1/\lambda - 1}\, dy = c_\lambda\, \lambda\, \Gamma((k+1)/\lambda),$$

where $\Gamma$ is the Euler Gamma function. Using the approximation $\Gamma(x+1) \sim [x]!$, we see that

$$m_k = \int_{\mathbb{R}} x^k \, d\mu(x) \sim [\tfrac{k}{\lambda}]!$$

so by Stirling's formula ($\log(k!) \sim k \log k$),

$$\limsup_{k\to\infty} \frac{1}{2k} (m_{2k})^{\frac{1}{2k}} \sim \limsup_{k\to\infty} \frac{1}{2k} e^{\frac{1}{\lambda}\log(2k/\lambda)} \sim \limsup_{k\to\infty} \frac{1}{2k} (2k/\lambda)^{\frac{1}{\lambda}} < \infty$$

if and only if $\lambda \geq 1$. We can deduce the following rule of thumb from the preceding argument: a distribution is uniquely determined by its moments as long as its tail is not heavier than the exponential $e^{-x}$.

**3.** By ex. 2.b), we only need to check that for any $k \geq 0$,

$$\int_{\mathbb{R}} x^k \, dF_n(x) \underset{n\to\infty}{\to} m_k,$$

where $m_{2k+1} = 0$ and $m_{2k} = \dfrac{(2k)!}{2^k\,(k!)}$. Let us therefore compute

$$\int_{\mathbb{R}} x^k \, dF_n(x) = \mathbb{E}(X_n^k) = \frac{1}{n^{k/2}} \, \mathbb{E}\left((Y_1 + ... + Y_n)^k\right)$$

Using the multinomial expansion, we obtain

$$\mathbb{E}(X_n^k) = \frac{1}{n^{k/2}} \sum_{\substack{j_1,...j_n \geq 0 \\ j_1+...+j_n=k}} \begin{pmatrix} k \\ j_1, ..., j_n \end{pmatrix} \mathbb{E}(Y_1^{j_1} \cdots Y_n^{j_n}).$$

Since the $Y_j$ are i.i.d. and $\mathbb{E}(Y_j^{2l+1}) = 0$ for all $l \geq 0$, it is easy to see that the above sum is zero if $k$ is odd. Let us therefore consider

$$\begin{aligned} \mathbb{E}(X_n^{2k}) &= \frac{1}{n^k} \sum_{\substack{j_1,...j_n \geq 0 \\ j_1+...+j_n=2k}} \begin{pmatrix} 2k \\ j_1, ..., j_n \end{pmatrix} \mathbb{E}(Y_1^{j_1} \cdots Y_n^{j_n}) \\ &= \frac{1}{n^k} \sum_{\substack{l_1,...l_n \geq 0 \\ l_1+...+l_n=k}} \begin{pmatrix} 2k \\ 2l_1, ..., 2l_n \end{pmatrix} \mathbb{E}(Y_1^{2l_1}) \cdots \mathbb{E}(Y_n^{2l_n}). \end{aligned}$$

Let us divide this sum in two parts:

$$\begin{aligned} \mathbb{E}(X_n^{2k}) &= \frac{1}{n^k} \sum_{\substack{l_1,...l_n \in \{0,1\} \\ l_1+...+l_n=k}} \begin{pmatrix} 2k \\ 2l_1, ..., 2l_n \end{pmatrix} \mathbb{E}(Y_1^{2l_1}) \cdots \mathbb{E}(Y_n^{2l_n}) \\ &\quad + \frac{1}{n^k} \sum_{\substack{\exists 1\leq i\leq n \,:\, l_i \geq 2 \\ l_1+...+l_n=k}} \begin{pmatrix} 2k \\ 2l_1, ..., 2l_n \end{pmatrix} \mathbb{E}(Y_1^{2l_1}) \cdots \mathbb{E}(Y_n^{2l_n}). \end{aligned}$$

We see that the first term on the right-hand side is equal to

$$\frac{1}{n^k} \begin{pmatrix} n \\ k \end{pmatrix} \frac{(2k)!}{2^k} 1 = \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{(2k)!}{2^k\,k!} \underset{n\to\infty}{\to} \frac{(2k)!}{2^k\,k!}.$$

The theorem will therefore be proved if we check that the second term on the right-hand side goes to zero as $n \to \infty$:

$$\frac{1}{n^k} \sum_{\substack{\exists 1 \le i \le n \,:\, l_i \ge 2 \\ l_1 + \ldots + l_n = k}} \binom{2k}{2l_1, ..., 2l_n} \mathbb{E}(Y_1^{2l_1}) \cdots \mathbb{E}(Y_n^{2l_n}) \le \frac{C^{2k}}{n^k} \sum_{\substack{\exists 1 \le i \le n \,:\, l_i \ge 2 \\ l_1 + \ldots + l_n = k}} \frac{(2k)!}{(2l_1)! \cdots (2l_n)!}$$

$$\le \frac{C^{2k}}{n^k}\, n \sum_{m=2}^{k} \sum_{\substack{l_1, \ldots, l_{n-1} \ge 0 \\ l_1 + \ldots + l_{n-1} = k-m}} \frac{(2k)!}{(2l_1)! \cdots (2l_{n-1})!(2m)!}$$

$$\le \frac{C^{2k}\,(2k)!}{n^{k-1}} \sum_{m=2}^{k} \sum_{\substack{l_1, \ldots, l_{n-1} \ge 0 \\ l_1 + \ldots + l_{n-1} = k-m}} 1 \quad = \quad \frac{C^{2k}\,(2k)!}{n^{k-1}} \sum_{m=2}^{k} \binom{n+k-m-2}{k-m}$$

$$\le \frac{C^{2k}\,(2k)!}{n^{k-1}} \sum_{m=2}^{k} (n+k-m-2)^{k-m} \le \frac{C^{2k}\,(2k)!\,k}{n^{k-1}} (n+k)^{k-2} \le \frac{C_k}{n} \underset{n \to \infty}{\to} 0. \qquad \square$$