Solutions 5

1. 0) The “starter: use integration by parts:

$$\Gamma(x+1) = \int_0^\infty t^x\, e^{-t}\, dt = 0 - \int_0^\infty x\, t^{x-1}\, (-e^{-t})\, dt = x\, \Gamma(x).$$

a) Let us compute (NB: the integral is taken over the set of $n \times n$ positive semi-definite matrices)

$$\mathbb{E}(\det W) = \int dW\, p_W(W)\, (\det W) = C_{n,m} \int dW\, (\det W)^{m-n+1}\, \exp(-\mathrm{Tr}(W)) = \frac{C_{n,m}}{C_{n,m+1}},$$

since $C_{n,m+1}\, (\det W)^{m-n+1}\, \exp(-\mathrm{Tr}(W))$ is a probability distribution (on the set of $n \times n$ positive semi-definite matrices). Now,

$$\frac{C_{n,m}}{C_{n,m+1}} = \frac{\prod_{j=1}^n \Gamma(m-j+2)}{\prod_{j=1}^n \Gamma(m-j+1)} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} = \frac{m!}{(m-n)!}.$$

b) If $m = n$, then

$$\mathbb{E}(1/\det W) = C_{n,n} \int dW\, (\det W)^{-1}\, \exp(-\mathrm{Tr}(W)),$$

which diverges, because of the singularity at zero (consider e. g. the case $n = 1$). We therefore need to assume at least that $m > n$. In this case, we have

$$\begin{aligned}
\mathbb{E}(1/\det W) &= C_{n,m} \int dW\, (\det W)^{m-n-1}\, \exp(-\mathrm{Tr}(W)) = \frac{C_{n,m}}{C_{n,m-1}} \\
&= \frac{\prod_{j=1}^n \Gamma(m-j)}{\prod_{j=1}^n \Gamma(m-j+1)} = \frac{\Gamma(m-n)}{\Gamma(m)} = \frac{(m-n-1)!}{(m-1)!},
\end{aligned}$$

which is indeed finite.

c) Simply notice that

$$\psi(x+1) = \frac{d}{dx}\,(\log\Gamma(x+1)) = \frac{d}{dx}\,(\log x + \log\Gamma(x)) = \frac{1}{x} + \psi(x).$$

d) Let us compute (with slight abuses of notation, which can be made precise)

$$\begin{aligned}
\mathbb{E}(\log\det W) &= C_{n,m} \int dW\, \log(\det W)\, (\det W)^{m-n}\, \exp(-\mathrm{Tr}(W)) \\
&= C_{n,m}\, \frac{\partial}{\partial m} \left(\int dW (\det W)^{m-n}\, \exp(-\mathrm{Tr}(W)) \right) \\
&= C_{n,m}\, \frac{\partial}{\partial m} \left(\frac{1}{C_{n,m}} \right) = \frac{\partial}{\partial m} \left(\log \frac{1}{C_{n,m}} \right) \\
&= \sum_{j=1}^n \frac{\partial}{\partial m}\, (\log\Gamma(m-j+1)) = \sum_{j=1}^n \psi(m-j+1) \\
&= \sum_{j=1}^n \left(\sum_{k=1}^{m-j} \frac{1}{k} - \gamma \right) = \sum_{j=1}^n \sum_{l=j+1}^m \frac{1}{l-j} - n\gamma.
\end{aligned}$$

2. a) First observe that

$$\frac{1}{\det(I+HH^*)} = \frac{1}{\pi^n}\int_{\mathbb{C}^n} dz\,\exp(-\|z\|^2 - z^*HH^*z) = \mathbb{E}_z(\exp(-z^*HH^*z)),$$

where $z \sim \mathcal{N}_{\mathbb{C}}(0, I_n)$. So by Fubini's theorem,

$$\mathbb{E}_H\left(\frac{1}{\det(I+HH^*)}\right) = \mathbb{E}_H(\mathbb{E}_z(\exp(-z^*HH^*z))) = \mathbb{E}_z(\mathbb{E}_H(\exp(-z^*HH^*z))).$$

Let h_j denote the j^{th} column of H: we have $z^*HH^*z = \sum_{j=1}^n z^*h_jh_j^*z = \sum_{j=1}^n h_j^*zz^*h_j$. Therefore,

$$\mathbb{E}_H\left(\frac{1}{\det(I+HH^*)}\right) = \mathbb{E}_z\left(\mathbb{E}_H\left(\prod_{j=1}^n \exp(-h_j^*zz^*h_j)\right)\right) = \mathbb{E}_z\left((\mathbb{E}_H(\exp(-h^*zz^*h)))^n\right),$$

by independence of the columns of H (h denotes here one of these columns). Using the equality given in the problem set in the opposite direction, we get the result:

$$\mathbb{E}_H\left(\frac{1}{\det(I+HH^*)}\right) = \mathbb{E}_z\left(\frac{1}{\det(I+zz^*)^n}\right) = \mathbb{E}_z\left(\frac{1}{(1+\|z\|^2)^n}\right).$$

b) The technique is the same. One simply needs to observe that

$$\begin{aligned}\frac{1}{\det(I+HH^*)^k} &= \frac{1}{\pi^{nk}}\int_{\mathbb{C}^{nk}} dz_1\ldots dz_k\,\exp\left(-\sum_{j=1}^k \|z_j\|^2 - z_j^*HH^*z_j\right)\\ &= \mathbb{E}_{z_1,\ldots,z_k}\left(\exp\left(-\sum_{j=1}^k z_j^*HH^*z_j\right)\right),\end{aligned}$$

where $z_1,\ldots,z_k$ are i.i.d.$\sim \mathcal{N}_{\mathbb{C}}(0, I_n)$ random vectors. This may be rewritten as

$$\frac{1}{\det(I+HH^*)^k} = \mathbb{E}_Z(\exp(-\mathrm{Tr}(Z^*HH^*Z))),$$

where Z is an $n \times k$ matrix with i.i.d.$\sim \mathcal{N}_{\mathbb{C}}(0,1)$ entries. In a similar manner, one also gets

$$\frac{1}{\det(I+ZZ^*)^m} = \mathbb{E}_H(\exp(-\mathrm{Tr}(H^*ZZ^*H))),$$

and

$$\begin{aligned}\mathbb{E}_H\left(\frac{1}{\det(I+HH^*)^k}\right) &= \mathbb{E}_H\mathbb{E}_Z(\exp(-\mathrm{Tr}(Z^*HH^*Z)))\\ &= \mathbb{E}_Z\mathbb{E}_H(\exp(-\mathrm{Tr}(H^*ZZ^*H))) = \mathbb{E}_Z\left(\frac{1}{\det(I+ZZ^*)^m}\right)\end{aligned}$$

by Fubini's theorem and the use of $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.