

## Solutions 4

1.a) This is the difficult one. By the class,  $H$  has the following joint eigenvalue distribution:

$$p(\lambda_1, \dots, \lambda_n) = C_n \exp\left(-\frac{1}{2} \sum_{j=1}^n \lambda_j^2\right) \prod_{j < k} |\lambda_k - \lambda_j|.$$

Since  $H$  is real symmetric, its singular values are given by  $\sigma_j = |\lambda_j|$ . The resulting joint distribution of the singular values is therefore given by

$$p(\sigma_1, \dots, \sigma_n) = C_n \exp\left(-\frac{1}{2} \sum_{j=1}^n \sigma_j^2\right) \sum_{a_1, \dots, a_n \in \{\pm 1\}} \prod_{j < k} |a_k \sigma_k - a_j \sigma_j|,$$

but developing further this expression is probably difficult. Note that for  $n = 2$ , we have

$$p(\sigma_1, \sigma_2) = C_2 \exp(-(\sigma_1^2 + \sigma_2^2)/2) \max(\sigma_1, \sigma_2)$$

and for  $n = 3$ , we have for the ordered singular values  $\sigma_1 \geq \sigma_2 \geq \sigma_3$ :

$$p(\sigma_1, \sigma_2, \sigma_3) = C_3 \exp(-(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)/2) \sigma_2 (\sigma_1^2 - \sigma_3^2).$$

Can we go further?

b) This one is relatively easy: by the class, the joint eigenvalue distribution of  $HH^T$  is given by

$$p(\lambda_1, \dots, \lambda_n) = C_{n,m} \exp\left(-\frac{1}{2} \sum_{j=1}^n \lambda_j\right) \prod_{j=1}^n \lambda_j^{\frac{m-n-1}{2}} \prod_{j < k} |\lambda_k - \lambda_j|$$

and we know that  $\lambda_j = \sigma_j^2$ , where  $\sigma_j$  are the singular values of  $H$ . The Jacobian of this change of variables is given by

$$J = 2^n \prod_{j=1}^n \sigma_j,$$

so

$$p(\sigma_1, \dots, \sigma_n) = C_{n,m} 2^n \exp\left(-\frac{1}{2} \sum_{j=1}^n \sigma_j^2\right) \prod_{j=1}^n \sigma_j^{m-n} \prod_{j < k} |\sigma_k^2 - \sigma_j^2|.$$

c) This one is straightforward. The singular values of  $H$  are the (positive) square roots of the eigenvalues of  $HH^T$ , which are all equal to 1, since  $HH^T = I$ , so the joint distribution of the singular values of  $H$  is

$$p(\sigma_1, \dots, \sigma_n) = \prod_{j=1}^n \delta_1(\sigma_j)$$

NB: this is obviously *not* a pdf.

2.a) We have

$$\prod_{j < k} (e^{i\theta_j} - e^{i\theta_k}) = \det \begin{pmatrix} 1 & \dots & 1 \\ e^{i\theta_1} & \dots & e^{i\theta_n} \\ \dots & \dots & \dots \\ e^{i(n-1)\theta_1} & \dots & e^{i(n-1)\theta_n} \end{pmatrix} = \det \left( \left\{ e^{i(j-1)\theta_k} \right\} \right).$$

Using the definition of  $K$ , we therefore obtain

$$\begin{aligned} p(\theta_1, \dots, \theta_n) &= C_n \left| \det \left( \left\{ e^{i(j-1)\theta_k} \right\} \right) \right|^2 = C_n \det \left( \left\{ e^{i(j-1)\theta_k} \right\}^* \left\{ e^{i(j-1)\theta_k} \right\} \right) \\ &= C_n \det \left( \{K(\theta_j, \theta_k)\} \right). \end{aligned}$$

b) Property (0) is clear, property (i) follows from the fact that  $K(\theta, \theta) = \sum_{l=0}^{n-1} 1 = n$  and property (ii) follows from

$$\int_0^{2\pi} K(\theta, \varphi) K(\varphi, \psi) d\varphi = \sum_{l,m=0}^{n-1} e^{il\theta} \overline{e^{im\psi}} \int_0^{2\pi} \overline{e^{il\varphi}} e^{im\varphi} d\varphi = 2\pi \sum_{l=0}^{n-1} e^{il\theta} \overline{e^{il\psi}} = 2\pi K(\theta, \psi),$$

c) By expanding the determinant with respect to the  $m^{\text{th}}$  column, we obtain

$$\begin{aligned} &\int_0^{2\pi} D_m(\theta_1, \dots, \theta_m) d\theta_m \\ &= \sum_{l=1}^m (-1)^{l+m} \int_0^{2\pi} K(\theta_l, \theta_m) \det \left( \{K(\theta_j, \theta_k)\}_{j \neq l, k \neq m} \right) d\theta_m \\ &= \int_0^{2\pi} K(\theta_m, \theta_m) d\theta_m D_{m-1}(\theta_1, \dots, \theta_{m-1}) + \sum_{l=1}^{m-1} (-1)^{l+m} \int_0^{2\pi} K(\theta_l, \theta_m) \\ &\quad \cdot \left( \sum_{l'=1}^{m-1} (-1)^{l'+m-1} K(\theta_m, \theta_{l'}) \det \left( \{K(\theta_j, \theta_k)\}_{j \neq l, m; k \neq l', m} \right) \right) d\theta_m, \end{aligned}$$

where we have expanded each sub-determinant in the sum with respect to the  $(m-1)^{\text{th}}$  row (which is the  $m^{\text{th}}$  row of the original matrix).

Using properties (i) and (ii) of part b), we therefore obtain

$$\begin{aligned} &\int_0^{2\pi} D_m(\theta_1, \dots, \theta_m) d\theta_m \\ &= 2\pi n D_{m-1}(\theta_1, \dots, \theta_{m-1}) - 2\pi \sum_{l,l'=1}^{m-1} (-1)^{l+l'} K(\theta_l, \theta_{l'}) \det \left( \{K(\theta_j, \theta_k)\}_{j \neq l, m; k \neq l', m} \right) \\ &= 2\pi n D_{m-1}(\theta_1, \dots, \theta_{m-1}) - 2\pi \sum_{l=1}^{m-1} D_{m-1}(\theta_1, \dots, \theta_{m-1}) \\ &= 2\pi (n - m + 1) D_{m-1}(\theta_1, \dots, \theta_{m-1}), \end{aligned}$$

where we have again used the formula for the determinant (in the other direction).

d) By the formula found for  $p(\theta_1, \dots, \theta_n)$  in part a) and repeated applications of Mehta's lemma (or induction), we find that

$$\begin{aligned} p(\theta, \varphi) &= C_n \int_{[0, 2\pi]^{n-2}} D_n(\theta, \varphi, \theta_3, \dots, \theta_n) d\theta_3 \cdots d\theta_n \\ &= C_n 2\pi \int_{[0, 2\pi]^{n-3}} D_{n-1}(\theta, \varphi, \theta_3, \dots, \theta_{n-1}) d\theta_3 \cdots d\theta_{n-1} \\ &= \dots = C_{n,2} D_2(\theta, \varphi) = C_{n,2} (K(\theta, \theta) K(\varphi, \varphi) - |K(\theta, \varphi)|^2) \end{aligned}$$

and

$$p(\theta) = C_{n,2} \int_0^{2\pi} D_2(\theta, \varphi) d\varphi = C_{n,2} 2\pi(n-1) D_1(\theta) = C_{n,1} K(\theta, \theta).$$

g) Since  $K(\theta, \theta) = n$  and  $\int_0^{2\pi} p(\theta) d\theta = 1$ , we obtain that  $C_{n,1} = \frac{1}{2\pi n}$ , therefore

$$p(\theta) = \frac{1}{2\pi}, \quad \theta \in [0, 2\pi],$$

and  $C_{n,2} = C_{n,1}/(2\pi(n-1)) = 1/(4\pi^2 n(n-1))$ , so

$$p(\theta, \varphi) = \frac{1}{4\pi^2 n(n-1)} (n^2 - |K(\theta, \varphi)|^2).$$

Since

$$|K(\theta, \varphi)| = \left| \sum_{l=0}^{n-1} e^{il(\theta-\varphi)} \right| = \left| \frac{e^{in(\theta-\varphi)} - 1}{e^{i(\theta-\varphi)} - 1} \right| = \left| \frac{\sin\left(\frac{n(\theta-\varphi)}{2}\right)}{\sin\left(\frac{\theta-\varphi}{2}\right)} \right|,$$

we finally obtain that

$$p(\theta, \varphi) = \frac{1}{4\pi^2} \frac{n}{n-1} \left( 1 - \frac{\sin\left(\frac{n(\theta-\varphi)}{2}\right)^2}{\left(n \sin\left(\frac{\theta-\varphi}{2}\right)\right)^2} \right).$$

NB: two lines above, we have used

$$\left| e^{i\theta} - 1 \right| = \left| e^{i\theta/2} (e^{i\theta/2} - e^{-i\theta/2}) \right| = |2 \sin(\theta/2)|.$$