Solutions 3

A.1. First, $p(a,b,c) = \frac{1}{2(\pi^{3/2})} \exp(-(a^2+2b^2+c^2)/2)$. Next, writing term by term the change of variables gives

$$\begin{aligned} a(\lambda,\mu,\theta) &= \lambda\cos^2\theta + \mu\sin^2\theta = \frac{\lambda+\mu}{2} + \frac{\lambda-\mu}{2}\cos(2\theta), \\ b(\lambda,\mu,\theta) &= -(\lambda-\mu)\sin\theta\cos\theta = -\frac{\lambda-\mu}{2}\sin(2\theta), \\ c(\lambda,\mu,\theta) &= \lambda\sin^2\theta + \mu\cos^2\theta = \frac{\lambda+\mu}{2} - \frac{\lambda-\mu}{2}\cos(2\theta), \end{aligned}$$

so the Jacobian is given by

$$\begin{aligned} J(\lambda,\mu,\theta) &= \det\begin{bmatrix} \cos^2\theta & \sin^2\theta & -(\lambda-\mu)\sin(2\theta) \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & -(\lambda-\mu)\cos(2\theta) \\ \sin^2\theta & \cos^2\theta & (\lambda-\mu)\sin(2\theta) \end{bmatrix} \\ &= (\lambda-\mu)\det\begin{bmatrix} \cos^2\theta & \sin^2\theta & -\sin(2\theta) \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & -\cos(2\theta) \\ \sin^2\theta & \cos^2\theta & \sin(2\theta) \end{bmatrix} = \lambda-\mu. \end{aligned}$$

Since $a^2+2b^2+c^2 = \mathrm{Tr}(H_1H_1^T) = \lambda^2+\mu^2$, we deduce that

$$p(\lambda,\mu,\theta) = \frac{1}{2(\pi^{3/2})}\, e^{-(\lambda^2+\mu^2)/2}\,|\lambda-\mu|.$$

Now, since $p(\lambda,\mu,\theta)$ is independent of θ, $p(\theta) = \frac{2}{\pi}$ on $[0,\frac{\pi}{2}]$ and

$$p(\lambda,\mu) = \frac{1}{4\sqrt{\pi}}\, e^{-(\lambda^2+\mu^2)/2}\,|\lambda-\mu|.$$

One can check that this is indeed a joint density function on $\mathbb{R}^2$ (i.e. $\iint_{\mathbb{R}^2} d\lambda\, d\mu\, p(\lambda,\mu) = 1$).

Since $\lambda+\mu = \mathrm{Tr}(H_1) = a+c$ and $\lambda\mu = \det(H_1) = ac-b^2$, we have

$$\mathbb{E}(\lambda+\mu) = 0 \quad \text{and} \quad \mathbb{E}(\lambda\mu) = -\frac{1}{2}.$$

The next computation requires some more work:

$$\begin{aligned} \mathbb{E}(\max\{\lambda,\mu\}) &= \iint_{\mathbb{R}^2} d\lambda\, d\mu\, \max\{\lambda,\mu\}\, p(\lambda,\mu) \\ &= \frac{1}{4\sqrt{\pi}} \iint_{\mathbb{R}^2} d\lambda\, d\mu\, \max\{\lambda,\mu\}\, |\mu-\lambda|\, e^{-(\lambda^2+\mu^2)/2} \\ &= \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}} d\lambda \int_{-\infty}^{\lambda} d\mu\, \lambda\,(\lambda-\mu)\, e^{-(\lambda^2+\mu^2)/2}, \end{aligned}$$

by symmetry of the function. Computing integrals (using integration by parts) gives

$$\mathbb{E}(\max\{\lambda,\mu\}) = \frac{\sqrt{\pi}}{2}.$$

Finally, a computation similar to the preceding gives

$$p(\lambda) = \frac{e^{-\lambda^2}}{2\sqrt{\pi}} + \frac{e^{-\lambda^2/2}}{2\sqrt{\pi}}\,\lambda \int_0^\lambda d\mu\, e^{-\mu^2/2}.$$

A.2. The matrix of eigenvectors of H_2 is fixed and given by

$$V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

So we have the change of variables

$$\begin{bmatrix} a & c \\ c & a \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

i.e. $a = \frac{1}{2}(\lambda + \mu)$ and $c = \frac{1}{2}(\lambda - \mu)$. Computing the Jacobian of this linear transformation gives

$$J(\lambda, \mu) = -\frac{1}{2}.$$

Since $p(a, c) = \frac{1}{2\pi} \exp\left(-\frac{a^2+c^2}{2}\right)$ and $2(a^2 + c^2) = \mathrm{Tr}(H_2 H_2^T) = \lambda^2 + \mu^2$, we obtain

$$p(\lambda, \mu) = p(a(\lambda, \mu), b(\lambda, \mu))\,|J(\lambda, \mu)| = \frac{1}{4\pi} \exp\left(-\frac{\lambda^2 + \mu^2}{4}\right),$$

i.e. λ and μ are i.i.d.$\sim \mathcal{N}_{\mathbb{R}}(0, 2)$ random variables.

Since $\lambda + \mu = \mathrm{Tr}(H_2) = 2a$ and $\lambda + \mu = \det(H_2) = a^2 - c^2$, we have

$$\mathbb{E}(\lambda + \mu) = 0 \quad \text{and} \quad \mathbb{E}(\lambda\mu) = 0.$$

Now, we also obtain by symmetry that

$$\begin{aligned} \mathbb{E}(\max\{\lambda, \mu\}) &= 2 \int_{\mathbb{R}} d\lambda \int_\lambda^\infty d\mu\, \mu\, p(\lambda, \mu) \\ &= 2 \int_{\mathbb{R}} d\lambda\, \frac{1}{4\pi}\, e^{-\lambda^2/4} \int_\lambda^\infty d\mu\, \mu\, e^{-\mu^2/4} \\ &= \frac{1}{\pi} \int_{\mathbb{R}} d\lambda\, e^{-\lambda^2/2} = \sqrt{\frac{2}{\pi}}. \end{aligned}$$

Finally, we trivially have

$$p(\lambda) = \frac{1}{\sqrt{4\pi}}\, e^{-\lambda^2/4}.$$

B.1. The direct computation of the eigenvalues of H_1 gives

$$\lambda_1 = \frac{a+c}{2} + \sqrt{\left(\frac{a-c}{2}\right)^2 + b^2} \quad \text{and} \quad \lambda_2 = \frac{a+c}{2} - \sqrt{\left(\frac{a-c}{2}\right)^2 + b^2}.$$

NB: we have written λ_1, λ_2 instead of λ, μ in order to point out that λ_1, λ_2 are ordered here ($\lambda_1 \geq \lambda_2$).

Noticing that $\frac{a+c}{2}$, $\frac{a-c}{2}$ and b are i.i.d.$\sim \mathcal{N}_{\mathbb{R}}(0, \frac{1}{2})$ random variables, we obtain that the random vector $(\lambda_!, \lambda_2)$ is distributed as $(x+y, x-y)$ where x and y are independent, $x \sim \mathcal{N}_{\mathbb{R}}(0, \frac{1}{2})$ and y is distributed as a Rayleigh random variable (i.e. as the modulus of a $\mathcal{N}_{\mathbb{C}}(0,1)$ random variable):

$$p(y) = 2y \, \exp(-y^2), \quad y \geq 0.$$

The joint distribution of (x, y) is given by

$$p(x, y) = \frac{1}{\sqrt{\pi}} \, e^{-(x^2+y^2)} \, (2y), \quad x \in \mathbb{R}, y \geq 0.$$

The Jacobian of the linear transformation $x = \frac{\lambda_1+\lambda_2}{2}$, $y = \frac{\lambda_1-\lambda_2}{2}$ is

$$J(\lambda_1, \lambda_2) = -\frac{1}{2}$$

and we notice that $x^2 + y^2 = \frac{\lambda_1^2+\lambda_2^2}{2}$. Therefore,

$$p(\lambda_1, \lambda_2) = \frac{1}{2\sqrt{\pi}} \, e^{-\frac{\lambda_1^2+\lambda_2^2}{2}} \, (\lambda_1 - \lambda_2), \quad \text{for } \lambda_1 \geq \lambda_2.$$

The joint distribution of the unordered eigenvalues (λ, μ) is therefore given by

$$p(\lambda, \mu) = \frac{1}{4\sqrt{\pi}} \, e^{-\frac{\lambda^2+\mu^2}{2}} \, |\lambda - \mu|, \quad \lambda, \mu \in \mathbb{R}.$$

B.2. Computing directly the eigenvalues of H_2 gives $\lambda = a + c$ and $\mu = a - c$. Since for any $\alpha, \beta \in \mathbb{R}$,

$$\alpha\lambda + \beta\mu = (\alpha + \beta) \, a + (\alpha - \beta) \, c$$

is a Gaussian random variable (being the sum of two independent Gaussian random variables), the vector (λ, μ) is a Gaussian random vector, with zero mean and covariance matrix given by

$$\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

We therefore conclude that λ and μ are i.i.d.$\sim \mathcal{N}_{\mathbb{R}}(0, 2)$ random variables.