Solutions 2

1. We need to show that for any $\Sigma_1, \Sigma_2 > 0$ and $\alpha \in (0,1)$, we have

$$\alpha \log\det(I + Q\Sigma_1^{-1}) + (1-\alpha) \log\det(I + Q\Sigma_2^{-1}) \geq \log\det(I + Q(\alpha\,\Sigma_1 + (1-\alpha)\,\Sigma_2)^{-1}).$$

Let therefore $Z_1 \sim \mathcal{N}_{\mathbb{C}}(0,\Sigma_1)$, $Z_2 \sim \mathcal{N}_{\mathbb{C}}(0,\Sigma_2)$ be independent, and let Θ be independent of both Z_1 and Z_2 such that $\mathbb{P}(\Theta = 1) = \alpha = 1 - \mathbb{P}(\Theta = 0)$. Let also

$$Z = \begin{cases} Z_1, & \text{if } \Theta = 1, \\ Z_2, & \text{if } \Theta = 0. \end{cases}$$

We have:

$$\begin{aligned} I(X; X+Z) &\leq I(X; X+Z, \Theta) = I(X;\Theta) + I(X; X+Z \,|\, \Theta) \\ &= 0 + \alpha\, I(X; X+Z \,|\, \Theta = 1) + (1-\alpha)\, I(X; X+Z \,|\, \Theta = 0) \\ &= \alpha\, I(X; X+Z_1) + (1-\alpha)\, I(X; X+Z_2) \\ &= \alpha \log\det(I + Q\Sigma_1^{-1}) + (1-\alpha) \log\det(I + Q\Sigma_2^{-1}) \end{aligned}$$

On the other hand,

$$\mathbb{E}(ZZ^*) = \alpha\,\mathbb{E}(ZZ^* \,|\, \Theta = 1) + (1-\alpha)\,\mathbb{E}(ZZ^* \,|\, \Theta = 0) = \alpha\,\Sigma_1 + (1-\alpha)\,\Sigma_2,$$

so

$$I(X; X+Z) \geq \log\det(I + Q(\alpha\,\Sigma_1 + (1-\alpha)\,\Sigma_2)^{-1}),$$

which concludes the proof.

2. a) By the Cauchy-Schwarz inequality, we obtain:

$$\left|\frac{1}{n}\mathrm{Tr}(A)\right| = \left|\frac{1}{n}\sum_{j=1}^n a_{jj}\right| \leq \frac{1}{n}\sqrt{n\sum_{j=1}^n |a_{jj}|^2} \leq \sqrt{\frac{1}{n}\sum_{j,k=1}^n |a_{jk}|^2} = \|A\|_2.$$

For $k \in \{1,\ldots,n\}$, we denote by $\delta^{(k)}$ the column vector whose components are given by $\delta_j^{(k)} = 1$ if $j = k$, 0 otherwise. We then have

$$|||A|||_2^2 \geq \max_{k\in\{1,\ldots,n\}} \|A\delta^{(k)}\|^2 \geq \frac{1}{n}\sum_{k=1}^n \|A\delta^{(k)}\|^2 = \frac{1}{n}\sum_{j,k=1}^n |a_{jk}|^2 = \|A\|_2^2.$$

b) We see that since $\|Ax\| \leq |||A|||\,\|x\|$ for all $x \in \mathbb{C}^n$,

$$|||AB|||_2 = \sup_{x\in\mathbb{C}^n:\|x\|=1} \|ABx\| \leq \sup_{x\in\mathbb{C}^n:\|x\|=1} |||A|||_2\,\|Bx\| = |||A|||_2\,|||B|||_2.$$

Next, let us denote by $b^{(k)}$ the k-th column vector of the matrix B (*i.e.*, $b_j^{(k)} = b_{jk}$); we have

$$\begin{aligned} \|AB\|_2^2 &= \frac{1}{n}\sum_{j,k=1}^n \left|\sum_{l=1}^n a_{jl}b_{lk}\right|^2 = \frac{1}{n}\sum_{j,k=1}^n \left|(Ab^{(k)})_j\right|^2 = \frac{1}{n}\sum_{k=1}^n \|Ab^{(k)}\|^2 \\ &\leq \frac{1}{n}\sum_{k=1}^n |||A|||_2^2\,\|b^{(k)}\|^2 = |||A|||_2^2\,\|B\|_2^2. \end{aligned}$$

Finally, by choosing A and B to be the "all ones" matrices, we obtain that

$$\|AB\|_2 = \sqrt{\frac{1}{n}\sum_{j,k=1}^{n} n^2} = n^{3/2} > n = \sqrt{n}\,\sqrt{n} = \|A\|_2\,\|B\|_2.$$

c) We have

$$|||A|||_2^2 = \sup_{x\in\mathbb{C}^n:\|x\|=1} x^* A^* A x,$$

and A^*A is diagonalizable (because it is Hermitian), so $A^*A = U^*DU$ for some unitary matrix U and $D = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_n^2)$. This implies that

$$|||A|||_2^2 = \sup_{x\in\mathbb{C}^n:\|x\|=1} x^*U^*DUx = \sup_{x\in\mathbb{C}^n:\|x\|=1} x^*Dx = \sup_{x\in\mathbb{C}^n:\|x\|=1} \sum_{j=1}^{n} \sigma_j^2 |x_j|^2 = \max_{j\in\{1,\ldots,n\}} \sigma_j^2.$$

Similarly,

$$\|A\|_2^2 = \frac{1}{n}\mathrm{Tr}(A^*A) = \frac{1}{n}\mathrm{Tr}(U^*DU) = \frac{1}{n}\mathrm{Tr}(D) = \frac{1}{n}\sum_{j=1}^{n}\sigma_j^2.$$

d) The condition allows to conclude only for $m = 1$ (and obviously, $m = 0$). In order to be able to conclude for any $m \geq 0$, we need to assume in addtion that there exists $C > 0$ such that for all $n \geq 1$,

$$|||A^{(n)}||| \leq C, \quad |||B^{(n)}||| \leq C.$$

From this, we indeed deduce that

$$\begin{aligned}
\left|\frac{1}{n}\mathrm{Tr}((A^{(n)})^m) - \frac{1}{n}\mathrm{Tr}((B^{(n)})^m)\right| &= \left|\frac{1}{n}\sum_{j=1}^{m} \mathrm{Tr}\left((B^{(n)})^{j-1}\,(A^{(n)} - B^{(n)})\,(A^{(n)})^{m-j}\right)\right| \\
&\leq \sum_{j=1}^{m} \left|\frac{1}{n}\mathrm{Tr}\left((A^{(n)})^{m-j}\,(B^{(n)})^{j-1}\,(A^{(n)} - B^{(n)})\right)\right| \\
&\leq \sum_{j=1}^{m} \|(A^{(n)})^{m-j}(B^{(n)})^{j-1}\,(A^{(n)} - B^{(n)})\|_2 \\
&\leq \sum_{j=1}^{m} |||(A^{(n)})^{m-j}|||_2\,|||(B^{(n)})^{j-1}|||_2\,\|A^{(n)} - B^{(n)}\|_2 \\
&\leq m\,C^{m-1}\,\|A^{(n)} - B^{(n)}\|_2 \underset{n\to\infty}{\to} 0.
\end{aligned}$$