Solutions 1

1. First of all, since λ is an eigenvalue of A, there exists $u \in \mathbb{C}^n$ such that $u \neq 0$ and $Au = \lambda u$. In particular, $u^* A u = \lambda\, u^* u$.

a1) If $A^* = A$, then
$$\lambda\, u^* u = u^* A u = u^* A^* u = (Au)^* u = (\lambda u)^* u = \overline{\lambda}\, u^* u,$$
i.e. $\lambda \in \mathbb{R}$, since $u^* u \neq 0$.

a2) If $A^* = -A$, then similarly,
$$\lambda\, u^* u = u^* A u = -u^* A^* u = -\overline{\lambda}\, u^* u,$$
i.e. $\lambda \in i\mathbb{R}$, since $u * u \neq 0$.

b1) If $a_{jk} \in \mathbb{R}$ for all $j, k \in \{1, \ldots, n\}$, then we have
$$\overline{Au} = \overline{\lambda u}, \quad \text{so} \quad A\overline{u} = \overline{\lambda}\, \overline{u}.$$
i.e. $\overline{\lambda}$ is also an eigenvalue of A.

b2) If $a_{jk} \in i\mathbb{R}$ for all $j, k \in \{1, \ldots, n\}$, then we have
$$\overline{Au} = \overline{\lambda u}, \quad \text{so} \quad -A\overline{u} = \overline{\lambda}\, \overline{u}.$$
i.e. $-\overline{\lambda}$ is also an eigenvalue of A.

c) If A is positive semi-definite, then $\lambda\, u^* u = u^* A u \geq 0$ by assumption, so $\lambda \geq 0$, since $u^* u > 0$.

d1) If $A^* A = I$, then
$$|\lambda|^2\, \|u\|^2 = \|\lambda u\|^2 = \|Au\|^2 = u^* A^* A u = u^* u, \quad \text{so } |\lambda| = 1.$$

d2) Notice that $(A^2)_{jk} = 1$ if $j + k = n$ or $j = k = n$, and 0 otherwise. Therefore, is is easy to infer that $A^4 = I$, so that whenever λ is an eigenvalue of A, we have $\lambda^4 = 1$, i.e. $\lambda \in \{1, i, -1, -i\}$.

e1) If $a_{jk} \geq 0$ for all $j, k \in \{1, \ldots, n\}$ and $Au = \lambda u$, then let $l = \text{argmax}_j |u_j|$. We have
$$|\lambda u_l| = \left| \sum_{k=1}^n a_{lk}\, u_k \right| \leq \sum_{k=1}^n a_{lk} |u_k| \leq \sum_{k=1}^n a_{lk} |u_l|.$$
Since $u \neq 0$, $|u_l| > 0$, so
$$|\lambda| \leq \sum_{k=1}^n a_{lk} \leq \max_{j \in \{1,\ldots,n\}} \sum_{k=1}^n a_{jk}.$$

e2) It is a direct consequence of e1), since $\sum_{k=1}^n a_{jk} = 1$ for all $j \in \{1, \ldots, n\}$.

f) Let again $l = \text{argmax}_j |u_j|$. $Au = \lambda u$ implies that

$$(\lambda - a_{ll})\, u_l = \sum_{k=1, k\neq l}^{n} a_{lk}\, u_k,$$

so

$$|\lambda - a_{ll}|\, |u_l| \leq \sum_{k=1, k\neq l}^{n} |a_{lk}|\, |u_k| \leq \sum_{k=1, k\neq l}^{n} |a_{lk}|\, |u_l|.$$

This in turn implies (since $|u_l| > 0$) that

$$|\lambda - a_{ll}| \leq \sum_{k=1, k\neq l}^{n} |a_{lk}|,$$

so

$$\lambda \in \bigcup_{j=1}^{n} B\left(a_{jj}, \sum_{k=1, k\neq j}^{n} |a_{jk}|\right).$$

2. $u^* A u \geq 0$ for all $u \in \mathbb{C}^n$ implies that

a) $a_{jj} \geq 0$ (take $u = e_j$).

b) $|\alpha|^2\, a_{jj} + \alpha\, \overline{\beta}\, a_{jk} + \overline{\alpha}\, \beta\, a_{kj} + |\beta|^2\, a_{kk} \geq 0$ (take $u = \alpha\, e_j + \beta\, e_k$).

Since $a_{jj}, a_{kk} \in \mathbb{R}$, choosing $\alpha = \beta = 1$ gives $a_{jk} + a_{kj} \in \mathbb{R}$, i.e., $\text{Im}(a_{kj}) = -\text{Im}(a_{jk})$. On the other hand, choosing $\alpha = i$ and $\beta = 1$ gives $i(a_{jk} - a_{kj}) \in \mathbb{R}$, i.e., $\text{Re}(a_{kj}) = \text{Re}(a_{jk})$. In total, this implies that $a_{kj} = \overline{a_{jk}}$, i.e., that $A = A^*$.

c) Taking $\beta = 1$ in the preceding inequality and α arbitrary gives

$$|\alpha|^2\, a_{jj} + 2\, \text{Re}(\alpha\, a_{jk}) + a_{kk} \geq 0.$$

Choosing $\alpha = r e^{-i \arg(a_{jk})}$ gives

$$r^2\, a_{jj} + 2\, r\, |a_{jk}| + a_{kk} \geq 0.$$

Since r is arbitrary, this implies that the discriminant of this second order equation is non-positive, i.e. that

$$|a_{jk}|^2 - a_{jj} a_{kk} \leq 0.$$