Random matrix theory : lecture 5

Preliminary : Haar "uniform" distribution

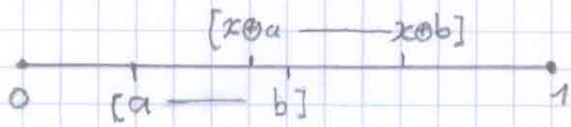
Theorem

Let G be a compact group (not necessarily commutative).
There exists a unique Borel probability distribution μ on G
such that $\mu(B) = \mu(g \cdot B) = \mu(B \cdot g) \quad \forall g \in G,\ B \in \mathcal{B}(G)$,
where
$$\begin{cases} g \cdot B := \{ h \in G : \exists g_0 \in B \text{ s.t. } h = g\, g_0 \} \\ B \cdot g := \{ h \in G : \exists g_0 \in B \text{ s.t. } h = g_0\, g \} \end{cases}$$
μ is called the Haar distribution on G.

Examples

0) Let $G = S(n)$ the group of permutations σ on $\{1..n\}$ equipped with the usual composition law $\sigma \circ \tau$.
Then $\mu(\{\sigma\}) = \frac{1}{n!}\ \forall \sigma$ is the Haar distribution on $S(n)$.

1) Let $G = [0,1[$ equipped with the addition modulo 1 $(x \oplus y)$.
Then $\mu([a,b]) = b - a$ is the Haar distribution on G
i.e. $\mu(x \oplus [a,b]) = \mu([a,b]) \quad \forall x, a, b \in [0,1[$



$$\int_0^1 f(x)\, d\mu(x) = \int_0^1 f(x)\, dx \quad , \quad f : [0,1[\to \mathbb{R}$$

× 2) Let $G = O(n) = \{ V \text{ } n\times n \text{ real matrix s.t. } VV^T = I \}$ (i.e. orthogonal matrix)

equipped with the standard matrix product.

Then the Haar distribution μ on G satisfies

× $\mu(V \cdot B) = \mu(B \cdot V) = \mu(B) \quad \forall V_{\text{orthogonal}},\ B \in \mathcal{B}(G)$

2b) Let $G = SO(n) = \{ V \in O(n) : \det V = +1 \}$ [rotations in $\mathbb{R}^n$]

(equipped again with the standard matrix product)

<u>NB</u>: $SO(2) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : 0 \le \theta < 2\pi \right\}$

is isomorphic to example 1.

3) Let $G = \mathcal{U}(n) = \{ U \text{ } n\times n \text{ complex matrix s.t. } UU^* = I \}$ (i.e. unitary matrix)

<u>NB</u>: $\mathcal{U}(1) = \{ e^{i\theta} : 0 \le \theta < 2\pi \}$

is also isomorphic to example 1

3b) Let $G = S\mathcal{U}(n) = \{ U \in \mathcal{U}(n) : \det U = +1 \}$

<u>Concrete example of Haar distribution</u> : $G = SO(3)$ [rotations in $\mathbb{R}^3$]

- Let $Z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $X(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{pmatrix}$
- Any $V \in SO(3)$ may be represented as $Z = X(\varphi_1)\, Z(\theta)\, X(\varphi_2)$

  for some $\varphi_1 \in [0, 2\pi]$, $\theta \in [0, \pi]$, $\varphi_2 \in [0, 2\pi]$ (Euler angles)
- × $\displaystyle\int_{SO(3)} f(V)\, d\mu(V) = \frac{1}{8\pi^2} \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\varphi_1 \int_0^{2\pi} d\varphi_2\, f(X(\varphi_1) Z(\theta) X(\varphi_2))$

  $(f : SO(3) \to \mathbb{R})$

Circular Orthogonal Ensemble (COE)

Let V be a $n \times n$ real orthogonal matrix picked "uniformly at random" i.e. according to the Haar distribution on $O(n)$.

Remarks:

- These matrices arise frequently in wireless commucations (as well as their sisters from the CUE).
- As already seen, there are only $\frac{n(n-1)}{2}$ free parameters in an orthogonal matrix V, so not all the entries of V can be independent. It is actually quite difficult to describe the joint distribution of the entries of V.

Question: how to pick V from the COE in practice?

1) Let $v_1 \ldots v_n$ be the columns of V:
   - pick v_1 uniformly on the unit sphere ($\|v_1\| = 1$)
   - pick v_2 uniformly on the set $\{v_2 \in \mathbb{R}^n : \|v_2\| = 1 \ \& \ v_2 \perp v_1\}$
   - and so on ... until v_n, which is fixed. (up to a $\pm$)

× 2) Pick H from the GOE; compute its eigenvectors; the resulting matrix V of eigenvectors is then a matrix from the COE.

3) Let H be a matrix with iid $\sim \mathcal{N}_{\mathbb{R}}(0,1)$ entries; perform the Gram-Schmidt decomposition of H; the resulting orthogonal matrix is again a matrix from the COE. [ref: Eaton, Multivariate Statistics]

<u>Joint eigenvalue distribution</u>

<u>NB</u>: V is orthogonal ($VV^T = V^TV = I$), so:

- its eigenvalues are located on the unit circle in $\mathbb{C}$,
× i.e. $|\lambda_j| = 1 \quad \forall j$ (but the λ_j are <u>not</u> real in general)
- V is normal ($VV^* = V^*V$), so it is unitarily
× diagonalizable (but <u>not</u> necessarily orthogonally diagonalizable)

Let us write $\lambda_j = e^{i\theta_j}$ with $\theta_j \in [0, 2\pi[$

Then $p(\theta_1 \ldots \theta_n) = \underset{\substack{\text{interpretation: } 1 \\ \text{(unif. dist.)}}}{C_n} \cdot \underbrace{\prod_{j<k} |e^{i\theta_k} - e^{i\theta_j}|}_{\text{Jacobian (cf. GOE)}}$ (no proof)

Circular Unitary Ensemble (CUE)

Let U be a $n \times n$ complex unitary matrix picked according to the Haar distribution on $U(n)$.

Similar remarks as above apply. Notice one difference: in a unitary matrix, there are $\underset{\text{choice for } u_1}{(2n-1)} + \underset{\text{choice for } u_2}{(2n-3)} + \ldots + 1 = n^2$ free real parameters

Again, since U is unitary ($UU^* = U^*U = I$), it is normal, therefore unitarily diagonalizable and its eigenvalues are of the form $\lambda_j = e^{i\theta_j}$, $\theta_j \in [0, 2\pi[$.

Joint eigenvalue distribution:

$$p(\theta_1, \ldots, \theta_n) = \underset{1,\text{ again}}{C_n} \cdot \underbrace{\prod_{j<k} |e^{i\theta_k - i\theta_j}|^2}_{\text{Jacobian (cf. GUE)}}$$

× Marginals:

$$p(\theta) = \frac{1}{2\pi} \quad \text{(uniform distribution on the circle)}$$

$$p(\theta, \varphi) = \frac{1}{4\pi^2} \cdot \frac{n}{n-1} \left(1 - \left(\frac{\sin\left(\frac{n(\theta-\varphi)}{2}\right)}{n \sin\left(\frac{\theta-\varphi}{2}\right)} \right)^2 \right)$$

[→ homework 3 (+ proof of Mehta's lemma in a simple case)]

Generalization and "physical" interpretation

All the random matrix ensembles seen so far fit into the general model:

$$p(\lambda_1 \ldots \lambda_n) = C_n \exp\left(-\sum_{j=1}^{n} V(\lambda_j)\right) \cdot \prod_{j<k} |\lambda_k - \lambda_j|^{\beta}$$

where V is a given function [NB: the corresponding distribution of entries is not known in general]

and $\beta = 1$ for real matrices, $\beta = 2$ for complex matrices

Observation: the Jacobian term $\prod_{j<k} |\lambda_k - \lambda_j|^{\beta}$ is very small whenever two eigenvalues are close to each other; i.e., on average, the eigenvalues tend to repel each other (& more in the complex case than in the real case)

Rewriting:

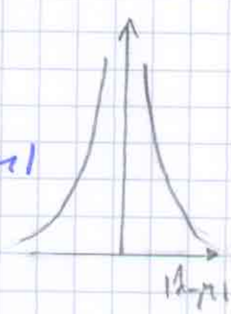
$$\underset{\uparrow}{p(\lambda_1 \ldots \lambda_n)} = C_n \cdot \exp\Bigg(-\underbrace{\Big(\underset{\uparrow}{\sum_{j=1}^{n} V(\lambda_j)} - \beta \underset{\uparrow}{\sum_{j<k} \log |\lambda_k - \lambda_j|}\Big)}_{\text{energy}}\Bigg)$$

Gibbs distribution — potential term — interaction term (repulsion)

$\lambda_1 \ldots \lambda_n$ = "particles" tending to minimize energy

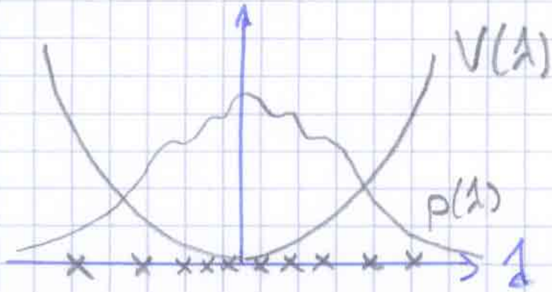
i.e., a) to find the minima of the potential V

b) coping with the repulsion term $-\beta \log |\lambda - \mu|$



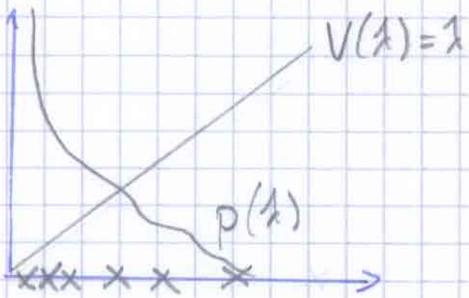
Examples

- **GOE or GUE**: $\lambda_j \in \mathbb{R}$ and $V(\lambda) = \frac{\lambda^2}{2}$

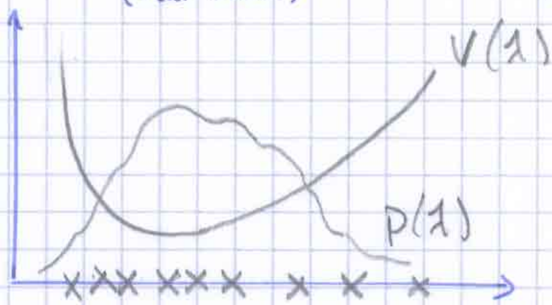


- **Real or complex Wishart ensemble**: $(m \geq n)$

$$\lambda_j \geq 0 \quad \text{and} \quad V(\lambda) = \underset{\text{(real case)}}{\frac{\lambda}{2} - \left(\frac{m-n-1}{2}\right)\log\lambda} \quad \text{or} \quad \underset{\text{(complex case)}}{\lambda - (m-n)\log\lambda}$$

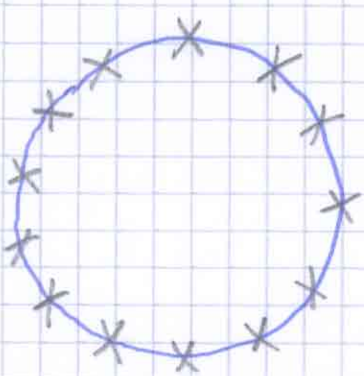


$m = n$ (complex case)



$m > n$ (complex case)

- **COE or CUE**: $|\lambda_j| = 1$, i.e. $\lambda_j = e^{i\theta_j}$ and $V(\lambda) \equiv 1$



uniform and regular distribution of the eigenvalues on the unit circle ($\neq$ iid points!)