Random matrix theory: lecture 3

Real Wishart Ensemble (non-std terminology)

Let H be a $n\times m$ real random matrix such that $\{h_{jk}, 1\le j\le n, 1\le k\le m\}$ are i.i.d. r.v. $\sim N_{\mathbb{R}}(0,1)$.
Let $W = HH^T$ ($n\times n$ matrix) and $\lambda_1 .. \lambda_n$ be the eigenvalues of W. (not those of H (!))

×

Remarks: • since W is symmetric $(W^T = HH^T = W)$ & non-negative definite, $(u^T W u = \|H^T u\|^2 \ge 0)$

$$\lambda_j \ge 0 \quad \forall 1\le j\le n$$

• if $m<n$, then $n-m$ eigenvalues are zero, since $\text{rank}(W) \le \text{rank}(H) \le \min(n,m) = m$

× In the case where $m<n$, there is therefore no joint eigenvalue density $p(\lambda_1,..,\lambda_n)$, but we may as well consider $\tilde{W} = H^T H$ and $\tilde{\lambda}_1 .. \tilde{\lambda}_m$ the eigenvalues of $\tilde{W}$, since the non-zero eigenvalues of W and $\tilde{W}$ coincide.

Proof, together with a linear algebra reminder:

Let A be a $n\times n$ matrix, B a $n\times m$ matrix

C a $m\times n$ matrix, D a $m\times m$ matrix

If A is invertible, then

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\cdot\det(D - CA^{-1}B)$$

If D is invertible, then

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D)\cdot\det(A - BD^{-1}C)$$

In particular, for any B $n\times m$ & C $m\times n$, we have

$$\det(I_n - BC) = \det\begin{pmatrix} I_n & B \\ C & I_m \end{pmatrix} = \det(I_m - CB)$$

Likewise, for any $\lambda \neq 0$, we have

$$\det(\lambda I_n - BC) = \det(\lambda I_m - CB)$$

[⚠ this fails for $\lambda = 0$]

i.e. BC and CB share the same non-zero eigenvalues.

In particular, so do $W = HH^T$ and $\widetilde{W} = H^T H$. #

Without loss of generality, we may therefore assume that the $n\times m$ matrices H are "fat" ($\underline{m \geq n}$).

The case of "tall" matrices H $(m<n)$ is handled by replacing H by H^T and "adding" $n-m$ zero eigenvalues for the matrix W.

Preliminary question:

What is the joint distribution of the entries of W?

(= question asked by Wishart in 1928)

a) joint distribution of the entries of H:

$$p(H) = \prod_{j,k=1}^{n,m} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{h_{jk}^2}{2}\right)$$
$$= C_{n,m} \exp\left(-\frac{1}{2} \sum_{j,k=1}^{n,m} h_{jk}^2\right)$$
$$\times \quad = C_{n,m} \exp\left(-\frac{1}{2} \operatorname{Tr}(HH^T)\right)$$

b) LQ-decomposition of H: $(m \geq n)$

Any $n \times m$ matrix H can be decomposed as

$$H = LQ$$

× where L is a $n \times n$ lower-triangular matrix

and Q is a $n \times m$ matrix such that $QQ^T = I_n$

[generalization of orthogonal matrix]

c) Choleski decomposition of W: $(m \geq n)$

Consequently, $W = HH^T = LQQ^TL^T = LL^T$

General strategy: $p(H) \to p(L) \to p(W)$

(NB: $\neq$ method adopted by Wishart)

d) joint distribution of the entries of L : $(H = LQ)$

× The Jacobian of the change of var. $H \mapsto (L, Q)$ is given by

× $$J_1(L, Q) = \prod_{j=1}^{n} \ell_{jj}^{m-j} \quad (\geq 0) \qquad (nm \text{ variables})$$

Therefore,

× $$p(L, Q) = C_{n,m} \exp\left(-\tfrac{1}{2} \operatorname{Tr}(LL^T)\right) \cdot \prod_{j=1}^{n} \ell_{jj}^{m-j} = p(L)$$

(this expression again does not depend on Q)

× <u>NB</u>: since $\operatorname{Tr}(LL^T) = \sum_{j=1}^{n} \ell_{jj}^2 + \sum_{k<j} \ell_{jk}^2$, the above means that:

- the entries of L are independent
- the off-diagonal entries ℓ_{jk} $(k<j)$ are Gaussian
- the diagonal entries ℓ_{jj} are χ^2_{m+1-j}

e) joint distribution of the entries of W: $(W = LL^T)$

× The Jacobian of the change of var. $W \mapsto L$ is given by

× $$J_2(L) = 2^n \prod_{j=1}^{n} \ell_{jj}^{n+1-j} \quad (\geq 0) \qquad \left(\tfrac{n(n+1)}{2} \text{ variables}\right)$$

Therefore,

× $$p(W) = \frac{p(L)}{J_2(L)} = \tilde{C}_{n,m} \cdot \exp\left(-\tfrac{1}{2} \operatorname{Tr}(LL^T)\right) \prod_{j=1}^{n} \ell_{jj}^{m-n-1}$$

note that $\prod_{j=1}^{n} \ell_{jj} = \det L = \sqrt{\det W}$ and $LL^T = W$, so finally

$$p(W) = \tilde{C}_{n,m} \cdot \exp\left(-\tfrac{1}{2} \operatorname{Tr} W\right) \cdot (\det W)^{\frac{m-n-1}{2}}$$

(≥ 0)

Final step: joint eigenvalue distribution $p(\lambda_1, \dots, \lambda_n)$

There exist a $n \times n$ orthogonal matrix V and $\Lambda = \mathrm{diag}(\lambda_1 \dots \lambda_n)$ such that $W = V \Lambda V^T$. The Jacobian of the change of var. $W \mapsto (\Lambda, V)$ is again given by

× $$J_3(\Lambda, V) = \prod_{j<k} (\lambda_j - \lambda_k)$$

So

$$p(\Lambda, V) = p(V \Lambda V^T) \cdot |J(\Lambda, V)|$$
$$= \tilde{C}_{n,m} \exp\left(-\frac{1}{2} \mathrm{Tr}\,\Lambda\right) \cdot (\det \Lambda)^{\frac{m-n-1}{2}} \cdot |J(\Lambda, V)|$$
$$= \tilde{C}_{n,m} \exp\left(-\frac{1}{2} \sum_{j=1}^{n} \lambda_j\right) \cdot \prod_{j=1}^{n} \lambda_j^{\frac{m-n-1}{2}} \cdot \prod_{j<k} |\lambda_k - \lambda_j|$$
$$= p(\lambda_1 \dots \lambda_n)$$

(since again the above expression does not depend on V; we therefore have the same conclusions as in the GOE case)

References: (finite-size analysis)

- Madan Lal Mehta, "Random matrices"
- Alan Edelman's PhD thesis (available on the web)
- R.J. Muirhead, "Aspects of multivariate statistical theory"
- A.Gupta - D. Nagar, "Matrix variate distributions"
- V.Girko, "Theory of random determinants"

Corresponding complex ensembles

Gaussian Unitary Ensemble (GUE)

Let H be a $n \times n$ Hermitian random matrix such that

- $\{h_{jk}, j \leq k\}$ are independent random variables ($\& \; h_{kj} = \overline{h_{jk}}$)
- $h_{jj} \sim N_{\mathbb{R}}(0,1)$, $h_{jk} \sim N_{\mathbb{C}}(0,1)$ $j<k$ ("circularly symmetric")

  [i.e. $\mathrm{Re}\, h_{jk}, \mathrm{Im}\, h_{jk} \sim N_{\mathbb{R}}(0,\frac{1}{2})$ indep.]

joint distribution of entries:

× $$p(H) = C_n \exp\left(-\frac{1}{2}\sum_{j,k=1}^{n} |h_{jk}|^2\right) = C_n \exp\left(-\frac{1}{2}\operatorname{Tr}(HH^*)\right)$$

Jacobian of the transformation $H = U \Lambda U^*$:

× $$J(\Lambda, U) = \prod_{j<k} (\lambda_k - \lambda_j)^2 \qquad (\underline{NB}: \lambda_j \in \mathbb{R})$$

resulting joint eigenvalue distribution:

$$p(\lambda_1 .. \lambda_n) = C_n \exp\left(-\frac{1}{2}\sum_{j=1}^{n} \lambda_j^2\right) \cdot \prod_{j<k} (\lambda_k - \lambda_j)^2$$

Complex Wishart Ensemble

Let H be a $n \times m$ random matrix ($m \geq n$) such that

$\{h_{jk}, j=1..n, k=1..m\}$ are i.i.d. r.v. $\sim N_{\mathbb{C}}(0,1)$

Let $W = HH^*$ and $\lambda_1 .. \lambda_n$ be its eigenvalues (≥ 0)

Then $$p(\lambda_1 .. \lambda_n) = C_{n,m} \exp\left(-\sum_{j=1}^{n} \lambda_j\right) \left(\prod_{j=1}^{n} \lambda_j^{m-n}\right) \prod_{j<k} (\lambda_k - \lambda_j)^2$$

$$\left(\& \; p(W) = C_{n,m} \exp(-\operatorname{Tr} W) \, (\det W)^{m-n}\right)$$