Random matrix theory: Lecture 24

Recall:
- $\mathcal{T} = \bigoplus_{n \geq 0} \mathcal{H}^{\otimes n}$, with $\mathcal{H} = \mathbb{C}^2$, basis (e_1, e_2)
- $\mathcal{A} = \{ a : \mathcal{T} \to \mathcal{T}$ linear bounded operator $\}$
- $\varphi(a) = \langle 1, a 1 \rangle$
- a_i, a_i^* creation and annihilation operators
- $A_i = a_i + a_i^*$ distributed according to the semi-circle distribution: $R_{A_i}(z) = z$
- A_1 and A_2 freely independent

Comment 1

$$R_{A_1+A_2}(z) = R_{A_1}(z) + R_{A_2}(z) = z + z = 2z$$

i.e. $A_1 + A_2$ is again distributed according to the semi-circle distribution (with a different variance).

i.e. the semi-circle distribution plays the same role in free probability as the Gaussian distribution in classical probability. The analogy goes on with the following theorem (turn the page)..

Free central limit theorem

Let $a_1, \dots, a_n, \dots$ be a sequence of freely independent random variables, identically distributed and such that $\varphi(a_1) = 0$ and $\varphi(a_1^2) = 1$.

Let μ_n be the distribution of $\frac{1}{\sqrt{n}}(a_1 + \dots + a_n)$.

Then μ_n converges to the semi-circle distribution as $n \to \infty$.

(i.e. $R_{\mu_n}(z) \xrightarrow[n\to\infty]{} z \quad \forall z \in \mathbb{C}$)

Proof idea:

1) For c a constant and A a random variable, $R_{cA}(z) = c R_A(cz)$:

$$\begin{cases} g_{cA}(z) = \varphi\left((cA - zI)^{-1}\right) = \frac{1}{c}\varphi\left(\left(A - \frac{z}{c}I\right)^{-1}\right) = \frac{1}{c} g_A\left(\frac{z}{c}\right) \\ \text{so } g_{cA}^{-1}(z) = c\, g_A^{-1}(cz) \quad \left[\frac{1}{c} g_A\left(\frac{z}{c}\right) = \zeta \text{ iff } c\, g_A^{-1}(c\zeta) = z\right] \\ \text{and } R_{cA}(z) = g_{cA}^{-1}(-z) - \frac{1}{z} = c R_A(cz). \end{cases}$$

2) $R_{\frac{1}{\sqrt{n}}(a_1+\dots+a_n)}(z) = \frac{1}{\sqrt{n}} R_{a_1+\dots+a_n}\left(\frac{z}{\sqrt{n}}\right)$ by (1)

(by free indep.) $= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} R_{a_j}\left(\frac{z}{\sqrt{n}}\right) = \sqrt{n}\, R_{a_1}\left(\frac{z}{\sqrt{n}}\right)$

× 3) expansion: $R_{a_1}(z) = \sum_{k\geq 0} c_k z^k$ $\quad \begin{cases} c_0 = \varphi(a_1) = 0 \\ c_1 = \varphi(a_1^2) - \varphi(a_1)^2 = 1 \end{cases}$

$\Rightarrow R_{a_1}(z) = z + o(z)$

i.e. $R_{\frac{1}{\sqrt{n}}(a_1+\dots+a_n)}(z) = \sqrt{n}\left(\frac{z}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)\right) = z + o(1)$ #

Comment 2

The fact that $A_1 = a_1 + a_1^*$ is distributed according to the semi-circle distribution can be put in relation with the following:

- Let $\mathcal{A}^{(n)} := \{ n\times n \text{ (deterministic) matrices} \}$ (with $n\to\infty$)

$$\varphi(A) := a_{11}^{(n)}$$

Catalan numbers!

$$A^{(n)} := \begin{pmatrix} 0 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 0 \end{pmatrix} \quad \text{deterministic Toeplitz matrix}$$

Then $\lim_{n\to\infty} \varphi\left((A^{(n)})^k \right) = \begin{cases} \frac{1}{\ell+1}\binom{2\ell}{\ell} & \text{if } k=2\ell \\ 0 & \text{if } k=2\ell+1 \end{cases}$

i.e. as $n\to\infty$, $A^{(n)}$ is distributed according to the semi-circle distribution (with respect to expectation φ)!

"Proof"

× $\varphi\left((A^{(n)})^k\right) = (A^{(n)})^k_{11} \xrightarrow[n\to\infty]{} \#$ Dyck paths of length k

$$= \begin{cases} c_\ell & \text{if } k=2\ell \\ 0 & \text{otherwise} \end{cases} \qquad \#$$

- Note that with such an expectation φ, changing only two entries in the Toeplitz matrix changes drastically the limit: Let $B^{(n)} := \begin{pmatrix} 0 & 1 & & 0 & 1 \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 1 & 0 & & 1 & 0 \end{pmatrix}$ [circulant matrix!]

× Then $\lim_{n\to\infty} \varphi\left((B^{(n)})^k\right) = \begin{cases} \binom{2\ell}{\ell} & \text{if } k=2\ell \\ 0 & \text{otherwise} \end{cases}$ arcsine distribution

Proof: use $b_{11}^{(n)} = \frac{1}{n}\operatorname{Tr}(B^{(n)})$

Free multiplicative convolution and S-transform

Let A be a non-commutative random variable such that $\varphi(A) \neq 0$.

Let
$$\begin{cases} \psi_A(z) := \sum_{k\geq 1} \varphi(A^k) z^k & \left[= \varphi((I - zA)^{-1}) = -\frac{1}{z} g_A\left(\frac{1}{z}\right) - 1 \right] \\ S_A(z) := \frac{z+1}{z} \underbrace{\psi_A^{-1}(z)}_{\text{inverse function}} \end{cases}$$

(*)

g_A: Stieltjes transform

Proposition:

If A_1, A_2 are freely independent and s.t. $\varphi(A_1) \neq 0$, $\varphi(A_2) \neq 0$,

then $S_{A_1 A_2}(z) = S_{A_1}(z) S_{A_2}(z)$

and the distribution of $A_1 A_2$ is called the free multiplicative convolution and is denoted as $\mu_{A_1 A_2} = \mu_{A_1} \boxtimes \mu_{A_2}$.

Proof idea:

Same technique as for the R-transform:

× Let $A_1 = (I + a_1) \cdot p(a_1^*)$ with $p(z)$ some polynomial s.t. $p(0) \neq 0$.

× Then $S_{A_1}(z) = \frac{1}{p(z)}$; similarly $S_{A_2}(z) = \frac{1}{q(z)}$

and $S_{A_1 A_2}(z) = \frac{1}{p(z) q(z)}$ --- "#"

× (*) Example: $A \sim$ "quarter circle", i.e. $z g_A(z)^2 + z g_A(z) + 1 = 0$

$$\Rightarrow z(\psi_A(z) + 1)^2 = \psi_A(z) \Rightarrow S_A(z) = \frac{1}{z+1}$$

Application to random matrices

- Let $A^{(n)}, B^{(n)}$ be $n \times n$ real symmetric independent random matrices with limiting eigenvalue distributions μ_A, μ_B.$^{(*)}$
  Let $V^{(n)}$ be orthogonal (Haar dist.) & indep. of both $A^{(n)}$ and $B^{(n)}$.
- Then $A^{(n)}$ and $V^{(n)} B^{(n)} (V^{(n)})^T$ are asymptotically free, so the limiting eigenvalue distribution of $A^{(n)} V^{(n)} B^{(n)} (V^{(n)})^T$ is given by $\mu_{AB} = \mu_A \boxtimes \mu_B$ and can be computed via the S-transform.

(*) such that $\int_{\mathbb{R}} x \, d\mu_A(x) \neq 0$ and $\int_{\mathbb{R}} x \, d\mu_B(x) \neq 0$