Random matrix theory: lecture 23

Reminder

- A, B free non-commutative random variables, with corresponding distributions $\mu_A := (m_k^A, k \geq 0)$, $\mu_B := (m_k^B, k \geq 0)$

$\Rightarrow$ the distribution of $A+B$ given by $\mu_{A+B} := (m_k^{A+B}, k \geq 0)$ is the free additive convolution of μ_A and μ_B (denoted as $\mu_{A+B} = \mu_A \boxplus \mu_B$), which can be computed using the R-transform: $R_{A+B}(z) = R_A(z) + R_B(z)$.

- $A^{(n)}, B^{(n)}$ (nxn real symmetric) independent random matrices with limiting eigenvalue distributions μ_A, μ_B

$\Rightarrow$ let $V^{(n)}$ be orthogonal (Haar distributed) & independent of both $A^{(n)}$ and $B^{(n)}$; then $A^{(n)}$ & $V^{(n)} B^{(n)} (V^{(n)})^T$ are asymptotically free so the limiting eigenvalue distribution of $A^{(n)} + V^{(n)} B^{(n)} (V^{(n)})^T$ is given by $\mu_{A+B} = \mu_A \boxplus \mu_B$ and can be computed via the R-transform.

Today's program: [ref: Haagerup, Thorbjörnsen]

- construction of free random variables
- proof of the R-transform additivity rule.

(- free <u>multiplicative</u> convolution)

Construction of free random variables

Preliminary:

$\times$ • An $n \times n$ matrix A "is" a linear application $\mathbb{C}^n \to \mathbb{C}^n$.
It is entirely determined by its action on the elements of any basis of $\mathbb{C}^n$ (e.g. $e_1, .., e_n$):

• More generally, let $\mathcal{T}$ be a Hilbert space with a countable basis (possibly infinite) and a be a linear and bounded operator $\mathcal{T} \to \mathcal{T}$.
Then a is entirely determined by its action on the basis elements of $\mathcal{T}$.

Example:

Let $\mathcal{H} := \mathbb{C}^2$, with basis (e_1, e_2)

$\mathcal{H}^{\otimes n}$:= tensor product, with basis $(e_{i_1} \otimes .. \otimes e_{i_n},\ i_1 .. i_n \in \{1,2\})$

(with the convention $\mathcal{H}^{\otimes 0} := \mathbb{C} \cdot 1$ with basis (1))

$\mathcal{T} := \bigoplus_{n \geq 0} \mathcal{H}^{\otimes n}$; $\mathcal{T}$ has the following countable basis:

$(1, e_1, e_2, e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2, e_1 \otimes e_1 \otimes e_1, \ldots)$

Interpretation: elements of $\mathcal{T}$ represent a physical system in a given state: n represents the number of particles, each of which is either in state e_1 or state e_2;
1 (corresponding to $n=0$) represents the empty state.

In the following, we will consider non-commutative random variables as linear and bounded operators $a : \mathcal{T} \to \mathcal{T}$.

We also define the expectation: $\varphi(a) := \langle 1, a 1 \rangle$

× (cf. $\varphi(A) = a_{11}$ in the matrix case)

× <u>NB</u>: • the scalar product $\langle \cdot , \cdot \rangle$ on $\mathcal{T}$ is "defined" by

$$\langle e_{i_1} \otimes .. \otimes e_{i_n}, e_{j_1} \otimes .. \otimes e_{j_m} \rangle = \begin{cases} 1 & \text{if } m=n,\ i_1 = j_1 \dots i_m = j_m \\ 0 & \text{otherwise} \end{cases}$$

• the corresponding norm on $\mathcal{T}$ is defined by

$$\| h \| := \sqrt{\langle h, h \rangle} \qquad h \in \mathcal{T}$$

• and $a : \mathcal{T} \to \mathcal{T}$ is bounded if $\exists k > 0$ such that

$$\| a h \| \leq k \| h \| \qquad \forall h \in \mathcal{T}$$

(note that an $n \times n$ matrix A always satisfies this condition)

<u>Examples</u> of non-commutative random variables on $\mathcal{T}$:

• "creation operator":

$$a_i (e_1 \otimes .. \otimes e_{i_n}) := \underset{\substack{\uparrow \\ \text{one more particle in state } i}}{e_i} \otimes e_{i_1} \otimes .. \otimes e_{i_n} \qquad i = 1, 2$$

• "annihilation operator":

$$a_i^* (e_{i_1} \otimes .. \otimes e_{i_n}) := \begin{cases} \overset{\text{one less particle in state } i}{e_{i_2}} \otimes \dots \otimes e_{i_n} & \text{if } i_1 = i \\ \underset{\substack{\uparrow \\ \dots \text{or even nothing!}}}{0} & \text{otherwise} \end{cases}$$

×

convention: $a_i^* 1 := 0$

Basic properties:

- $\forall h_1, h_2 \in T$, $\langle a_i^* h_1, h_2 \rangle = \langle h_1, a_i h_2 \rangle$; a_i^* = dual of a_i

Proof: check this for basis elements:

$$\langle a_i^* e_{i_1} \otimes .. \otimes e_{i_n}, e_{j_1} \otimes .. \otimes e_{j_m} \rangle =$$

$$\stackrel{?}{=} \langle e_{i_1} \otimes .. \otimes e_{i_n}, a_i e_{j_1} \otimes .. \otimes e_{j_m} \rangle$$

i.e. $\delta_{i,i_1} \cdot \langle e_{i_2} \otimes .. \otimes e_{i_n}, e_{j_1} \otimes .. \otimes e_{j_m} \rangle$

$$\stackrel{?}{=} \langle e_{i_1} \otimes .. \otimes e_{i_n}, e_i \otimes e_{j_1} \otimes .. \otimes e_{j_m} \rangle$$

Both sides are equal to 1 iff $n = m+1$, $i = i_1$, $i_2 = j_1 \dots, i_n = j_m$ #

- $a_i^* a_i = I$: $a_i^* a_i e_{i_1} \otimes .. \otimes e_{i_n} = a_i^* e_i \otimes e_{i_1} \otimes \dots \otimes e_{i_n}$
$= e_{i_1} \otimes .. \otimes e_{i_n}$ #

- $a_i^* a_j = 0$ if $i \neq j$: clear, since a_j creates a particle in state j; so a_i^* annihilates the whole system #

Proposition

Let $p(x,y)$, $q(x,y)$ be any two polynomials.

Then $p(a_1, a_1^*)$ and $q(a_2, a_2^*)$ are freely independent.

Proof:

- Since $a_1^* a_1 = I$, $p(a_1, a_1^*)$ may always be written as a sum of terms of the form $a_1^k (a_1^*)^l$ with $k+l>0$ (and the identity term). Same is true for $q(a_2, a_2^*)$.

- Let us therefore check the free independence of terms of the form $a_1^k (a_1^*)^\ell$ and $b_1^k (b_1^*)^\ell$ with $k+\ell>0$:

- first note that: $\varphi\left(a_i^k (a_i^*)^\ell\right) = \langle 1, a_i^k (a_i^*)^\ell 1 \rangle$

$$= \langle (a_i^*)^k 1, (a_i^*)^\ell 1 \rangle = 0 \text{ since at least } k>0 \text{ or } \ell>0$$

(recall that a_i^* = annihilation operator)

- there remains to show therefore that:

$$\varphi\left(a_1^{k_1}(a_1^*)^{\ell_1} \cdot a_2^{k_2}(a_2^*)^{\ell_2} \cdots\cdots a_1^{k_{2m-1}}(a_1^*)^{\ell_{2m-1}} a_2^{k_{2m}}(a_2^*)^{\ell_{2m}}\right) = 0$$

suppose it is not the case; then $k_1 = 0$, otherwise

$$\varphi(a_1^{k_1} \ldots) = \langle 1, a_1^{k_1} \ldots 1 \rangle = \langle (a_1^*)^{k_1} 1, \ldots 1 \rangle = 0$$

now, $k_1 + \ell_1 > 0 \Rightarrow \ell_1 > 0$, so $k_2 = 0$, otherwise

$$(a_1^*)^{\ell_1} a_2^{k_2} h = 0 \quad \forall h \quad \Rightarrow \varphi(\ldots) = 0$$

now $k_2 + \ell_2 > 0 \Rightarrow \ell_2 > 0$, so $k_3 = 0$ etc...

Finally, (by induction) $k_1 = k_2 = \ldots = k_{2m} = 0$ and

$$\varphi(\ldots) = \varphi\left((a_1^*)^{\ell_1}(a_2^*)^{\ell_2} \cdots (a_1^*)^{\ell_{2m-1}}(a_2^*)^{\ell_{2m}}\right) = 0:$$

contradiction.

so $\varphi(\ldots)$ must be equal to zero, and therefore $p(a_1, a_1^*)$ and $q(a_2, a_2^*)$ are indeed freely independent. #

Now that we know that free non-commutative random variables indeed exist, let us derive the additivity rule for the R-transform.

Proposition

Let $p(x)$ be a polynomial and $A_1 = a_1 + p(a_1^*)$.

Then $R_{A_1}(z) = p(z)$. [Example: $A_1 = a_1 + a_1^* \to R(z) = z$ semi-circle dist!]

Recall: • the distribution of A_1 is given by $m_k^{A_1} = \varphi(A_1^k)$

• its Stieltjes transform is given by $g_{A_1}(z) = \varphi((A_1 - zI)^{-1})$.

• its R-transform is given by $R_{A_1}(z) = g_{A_1}^{-1}(-z) - \frac{1}{z}$.

Proof:

× Let $\omega_z := \sum_{n\geq 0} z^n e_1^{\otimes n} = 1 + z e_1 + z^2 e_1 \otimes e_1 + \dots \in \mathcal{T}$ $(|z|<1)$

Then $a_1^* \omega_z = \sum_{n\geq 1} z^n e_1^{\otimes(n-1)} = z \sum_{n\geq 0} z^n e_1^{\otimes n} = \underset{\substack{\uparrow \\ \text{scalar multiplication!}}}{z}\,\omega_z$

so $p(a_1^*)\omega_z = p(z)\,\omega_z$

Also, $a_1 \omega_z = \sum_{n\geq 0} z^n e_1^{\otimes(n+1)} = \frac{1}{z} \sum_{n\geq 1} z^n e_1^{\otimes n} = \frac{1}{z}(\omega_z - 1)$

and $A_1 \omega_z = \frac{1}{z}(\omega_z - 1) + p(z)\omega_z = \left(\frac{1}{z} + p(z)\right)\omega_z - \frac{1}{z} 1$

i.e. $\left(A_1 - \frac{1}{z} - p(z)\right)\omega_z = -\frac{1}{z} 1$ or $\left(A_1 - \frac{1}{z} - p(z)\right)^{-1} 1 = -z\,\omega_z$

$\Rightarrow \underbrace{\varphi\left(\left(A_1 - \frac{1}{z} - p(z)\right)^{-1}\right)} = -z \langle 1, \omega_z \rangle = -z$

$= g_{A_1}\left(\frac{1}{z} + p(z)\right)$ so $g_{A_1}^{-1}(-z) = \frac{1}{z} + p(z)$ and $R(z) = p(z)$ #

Theorem

Let $p(x)$, $q(x)$ be any two polynomials $\left(\begin{array}{l}\text{may be extended to} \\ p, q \text{ analytic functions}\end{array}\right)$

and $A_1 = a_1 + p(a_1^*)$, $A_2 = a_2 + q(a_2^*)$

× Then A_1 and A_2 are freely independent

and $R_{A_1+A_2}(z) = R_{A_1}(z) + R_{A_2}(z)$.

Proof:

× One has to prove that $R_{A_1+A_2}(z) = p(z) + q(z)$.

Let $\varrho_z := \sum_{n \geq 0} z^n (e_1 + e_2)^{\otimes n} \qquad |z| < \frac{1}{2}$

Then $(a_1 + a_2)\varrho_z = \sum_{n \geq 0} z^n (e_1 + e_2)^{\otimes(n+1)} = \frac{1}{z}(\varrho_z - 1)$

$$a_1^* \varrho_z = \sum_{n \geq 0} z^n a_1^* (a_1 + a_2)^{\otimes n} 1$$

$$= \sum_{n \geq 1} z^n (a_1 + a_2)^{\otimes(n-1)} 1 \quad \text{since } a_1^* a_1 = I,\ a_1^* a_2 = 0$$

$$= \sum_{n \geq 1} z^n (e_1 + e_2)^{\otimes(n-1)} = z \varrho_z$$

Similarly, $p(a_1^*)\varrho_z = p(z)\varrho_z$; $a_2^* = z\varrho_z$, $q(a_2^*)\varrho_z = q(z)\varrho_z$

So $(A_1 + A_2)\varrho_z = \frac{1}{z}(\varrho_z - 1) + (p(z) + q(z))\varrho_z$

$\left(A_1 + A_2 - \frac{1}{z} - p(z) - q(z)\right)\varrho_z = -\frac{1}{z} 1$

$\left(A_1 + A_2 - \frac{1}{z} - p(z) - q(z)\right)^{-1} 1 = -z\varrho_z$

$g_{A_1+A_2}\left(\frac{1}{z} + p(z) + q(z)\right) = -z \quad \text{since } \langle 1, \varrho_z \rangle = 1$

i.e. $g_{A_1+A_2}^{-1}(-z) = \frac{1}{z} + p(z) + q(z)$

and $R_{A_1+A_2}(z) = g_{A_1+A_2}^{-1}(-z) - \frac{1}{z} = p(z) + q(z)$ #