Random matrix theory: Lecture 21

Gaussian random matrices and free probability

We have already seen that the following result holds (lecture 16):

- Let $A^{(n)} := \operatorname{diag}(a_1 .. a_n)$, with $a_j \in \mathbb{R}$ (deterministic), be such that $F_n^A(t) := \frac{1}{n} \# \{1 \le j \le n : a_j \le t\} \xrightarrow[n\to\infty]{} F^A(t)$, with corresponding Stieltjes transform $g_A(z)$.
- Let H be a $n\times n$ real symmetric matrix with iid $\sim N_{\mathbb{R}}(0,1)$ entries in the upper triangular part, and $H^{(n)} = \frac{1}{\sqrt{n}} H$.
(GOE model)
- Let $B^{(n)} := A^{(n)} + H^{(n)}$ and $\lambda_j^{(n)}$ be the e.v. of $B^{(n)}$. Then
$$F_n^B(t) := \frac{1}{n} \# \{ j : \lambda_j^{(n)} \le t \} \xrightarrow[n\to\infty]{} F^B(t) \quad \text{a.s.}$$
whose Stieltjes transform $g_B(z)$ satisfies $\boxed{g_B(z) = g_A(z + g_B(z))}$

Generalizations of this result:

- The result still holds if $A^{(n)}$ is deterministic and real symmetric with eigenvalues $a_1^{(n)} .. a_n^{(n)}$ and $F_n^A(t) = \frac{1}{n} \#\{j : a_j^{(n)} \le t\} \xrightarrow[n\to\infty]{} F^A(t)$

Proof:

$\exists\, O^{(n)}$ orthogonal and $D^{(n)} = \operatorname{diag}(a_1^{(n)} .. a_n^{(n)})$ s.t. $A^{(n)} = O^{(n)} D^{(n)} (O^{(n)})^T$

× $\Rightarrow B^{(n)} = O^{(n)} \underbrace{\Big(D^{(n)} + \underbrace{(O^{(n)})^T H^{(n)} O^{(n)}}_{\text{same dist. as } H^{(n)}} \Big)}_{\text{same e.v. as } B^{(n)}} (O^{(n)})^T$ #

see lecture 2

- The result still holds if $A^{(n)}$ is <u>random</u> and <u>independant</u> of $H^{(n)}$ with $F_n^A(t) \rightarrow F^A(t)$ a.s.

<u>Proof</u>:

× Conditioned on $A^{(n)}$, the result holds since $F_n^A(t) \rightarrow F^A(t)$ a.s. and F^A is deterministic #

<u>Remark</u>:

The result still holds for H with non-Gaussian entries, (i.e. non-orthogonally invariant) but this requires further work.

In lecture 16, we have also seen a result of the same flavor:

- Let $A^{(n)}$ be real symmetric & independent of H such that $F_n^A(t) \xrightarrow[n\to\infty]{} F^A(t)$ a.s., with Stieltjes transform $g_A(z)$.
- Let H be $n \times n$ with iid $\sim N_{\mathbb{R}}(0,1)$ entries and $W^{(n)} = \frac{1}{n} H H^T$.
- Let $B^{(n)} = A^{(n)} + W^{(n)}$. Then $F_n^B(t) \xrightarrow[n\to\infty]{} F^B(t)$ a.s., whose Stieltjes transform $g_B(z)$ satisfies

$$\boxed{g_B(z) = g_A\left(z - \frac{1}{1+g_B(z)}\right)}$$

Question

Is there a general rule for computing the limiting eigenvalue distribution of the sum of two independent random matrices $A^{(n)} + B^{(n)}$?

Answer 1

- A particular case of independent random matrices are deterministic matrices; and in this case, we know that there is no simple rule for computing the eigenvalues of $A^{(n)} + B^{(n)}$ from the eigenvalues of $A^{(n)}$ and $B^{(n)}$ separately, mainly because of the fact that they do not share the same eigenvectors in general.
- Moreover, even in the case where $A^{(n)}$ and $B^{(n)}$ share the same eigenvectors (when they are both diagonal, or both circulant, e.g.), everything is possible regarding the limiting eigenvalue distribution of $A^{(n)} + B^{(n)}$.

Example:

- Let $A^{(n)} = \mathrm{diag}\left(\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, \frac{n}{n}\right) = B^{(n)}$.

  Then the limiting eigenvalue distribution of both $A^{(n)}$ and $B^{(n)}$ is the uniform distribution on $[0,1]$.

  Also, $A^{(n)} + B^{(n)} = \mathrm{diag}\left(\frac{2}{n}, \frac{4}{n} \dots, \frac{2n}{n}\right)$ has for limiting eigenvalue distribution the uniform distribution on $[0,2]$.

- Let now $\widetilde{B}^{(n)} = \mathrm{diag}\left(\frac{n}{n}, \frac{n-1}{n}, \dots \frac{2}{n}, \frac{1}{n}\right)$

  Then the limiting eigenvalue distribution of $\widetilde{B}^{(n)}$ is also the uniform distribution on $[0,1]$, but $A^{(n)} + \widetilde{B}^{(n)} = \mathrm{diag}\left(\frac{n+1}{n}, \frac{n+1}{n}, \dots, \frac{n+1}{n}\right)$ has for limiting eigenvalue distribution the Dirac distribution at point $x = 1$.

In order to find a general rule for the limiting eigenvalue distribution of sums of random matrices, we need therefore to find a more restrictive condition than the independence of $A^{(n)}$ and $B^{(n)}$.

Important observation

When dealing with distributions of (eigenvalues of) random matrices, the framework of classical probability is not the best one, since any two classical random variables X and Y commute: $XY = YX$, but the same is not true for random matrices.

$\Rightarrow$ Non-commutative probability

Let $\mathcal{A}$ be the set of $n \times n$ $\underbrace{\text{matrices}}^{\text{(real)}}$; $\mathcal{A}$ is a non-commutative algebra, with addition $A+B$, multiplication $A \cdot B$ and unit element $A = I$. (*)

Def: an expectation on $\mathcal{A}$ is an application $\varphi : \mathcal{A} \to \mathbb{R}$ s.t.

- φ is linear: $\varphi(A + cB) = \varphi(A) + c\,\varphi(B)$
- $\varphi(I) = 1$
- $\varphi(A) \geq 0$ if $A \geq 0$

Examples:

- $\varphi(A) := \frac{1}{n} \operatorname{Tr} A$
- $\varphi(A) := a_{11}$

(*) and matrices are called non-commutative random variables.

Remarks:

- The set of classical random variables also forms an algebra, which is moreover commutative.
- So far, non-commutative random variables are $n \times n$ deterministic matrices (random matrices will come later).

What is the distribution of a non-commutative r.v.?

- The "distribution" of a matrix A is defined through its moments: $m_k = \varphi(A^k)$, $k \geq 0$, but there is in general no corresponding classical distribution μ_A on $\mathbb{R}$.
- For $\varphi(A) = \frac{1}{n} \cdot \operatorname{Tr} A$ and A real symmetric, there is:

$$\mu_A = \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_j^A}, \quad \text{where } \lambda_j^A = \text{e.v. of } A$$

$$\Rightarrow \begin{cases} m_k^A = \int_{\mathbb{R}} x^k \, d\mu_A(x) = \frac{1}{n} \sum_{j=1}^{n} (\lambda_j^A)^k = \frac{1}{n} \operatorname{Tr}(A^k) = \varphi(A^k). \\ g_A(z) = \int_{\mathbb{R}} \frac{1}{x-z} \, d\mu_A(z) = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{\lambda_j^A - z} \end{cases}$$

$$= \frac{1}{n} \operatorname{Tr}\left((A - zI)^{-1}\right) = \varphi\left((A - zI)^{-1}\right), \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

- Note moreover that in this case, we have

$$\varphi(AB) = \frac{1}{n} \operatorname{Tr}(AB) = \frac{1}{n} \operatorname{Tr}(BA) = \varphi(BA).$$