Random matrix theory: Lecture 20

Matrix inequalities: proofs from information theory

[ref: Dembo-Cover-Thomas, Diggavi-Cover]

Hadamard's inequality

Let $X=(X_1 .. X_n)$ be a complex random vector $\left(\begin{array}{c}\text{assuming a}\\ \text{joint pdf exists}\end{array}\right)$.

Then $h(X_1 .. X_n) \le \sum_{j=1}^n h(X_j)$

(obtained by the chain rule and the fact that conditioning reduces entropy)

Take $X \sim N_{\mathbb{C}}(0, A)$, with $A > 0$. The above inequality reads: $\log\det(\pi e A) \le \sum_{j=1}^n \log(\pi e\, a_{jj})$

i.e. $\det A \le \prod_{j=1}^n a_{jj}$ #

Fischer's inequality

Same idea: $h(X_1 .. X_m, X_{m+1} .. X_{m+n}) \le h(X_1 .. X_m) + h(X_{m+1} .. X_{m+n})$

Take $X \sim N_{\mathbb{C}}\left(0, \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}\right)$ with $A>0$ $m\times m$, $C>0$ $n\times n$

and $\underset{(m\times n)}{B}$ such that $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} > 0$. The above reads:

$$\log\det\left(\pi e \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}\right) \le \log\det(\pi e A) + \log\det(\pi e C)$$

i.e. $\det\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \le \det A \cdot \det C$ #

The application $A \mapsto \log\det A$ is concave on $\{A > 0\}$:

Let $X \sim N_{\mathbb{C}}(0, A)$, $Y \sim N_{\mathbb{C}}(0, B)$ be independent

Let Θ be indep. of both X and Y and be such that

$$\mathbb{P}(\Theta = 1) = \alpha = 1 - \mathbb{P}(\Theta = 0)$$

Let also $Z = \begin{cases} X & \text{if } \Theta = 1, \\ Y & \text{if } \Theta = 0. \end{cases}$ (NB: Z is not Gaussian)

(1) Then $h(Z) \geq h(Z \mid \Theta) = \alpha\, h(Z \mid \Theta = 1) + (1-\alpha)\, h(Z \mid \Theta = 0)$

$$= \alpha\, h(X) + (1-\alpha)\, h(Y)$$

(2) Now, $h(Z) \leq h(Z_G)$, where Z_G is Gaussian with the same covariance matrix Q as Z:

$$Q = \mathbb{E}(Z Z^*) = \alpha\, \mathbb{E}(X X^*) + (1-\alpha)\, \mathbb{E}(Y Y^*)$$

$$= \alpha A + (1-\alpha) B$$

(1+2) $\Rightarrow$ $\alpha \log\det(\pi e A) + (1-\alpha) \log\det(\pi e B)$

$$\leq \log\det\left(\pi e (\alpha A + (1-\alpha) B)\right)$$

i.e. $\alpha \log\det(A) + (1-\alpha) \log\det B \leq \log\det(\alpha A + (1-\alpha) B)$

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The application $A \mapsto \log\det(I + BA^{-1}B^*)$ is convex

Lemma: "The Gaussian noise is the worst noise" (without proof)

Let $X \sim N_{\mathbb{C}}(0, Q)$ and Z, Z_G be random vectors independent of X, with the same covariance matrix A. If $Z_G \sim N_{\mathbb{C}}(0, A)$, then $I(X; X+Z) \geq I(X; X+Z_G)$.

Let now $Z_1 \sim N_{\mathbb{C}}(0, A_1)$ and $Z_2 \sim N_{\mathbb{C}}(0, A_2)$ be independent and Θ be independent of both Z_1 & Z_2 and such that $\mathbb{P}(\Theta = 1) = \alpha = 1 - \mathbb{P}(\Theta = 0)$. Let also $Y = \begin{cases} X + Z_1 & \text{if } \Theta = 1 \\ X + Z_2 & \text{if } \Theta = 0 \end{cases}$

(1) Then $I(X;Y) \leq I(X;Y,\Theta) = \overbrace{I(X;\Theta)}^{=0} + I(X;Y|\Theta)$

$$= \alpha I(X;Y|\Theta=1) + (1-\alpha) I(X;Y|\Theta=0)$$

$$= \alpha I(X;X+Z_1) + (1-\alpha) I(X;X+Z_2)$$

$$= \alpha \log\det(I + QA_1^{-1}) + (1-\alpha)\log\det(I + QA_2^{-1})$$

(2) Also, $I(X;Y) \geq I(X;Y_G)$ where $Y_G = X + Z_G$ and $Z_G \sim N_{\mathbb{C}}(0, A)$ with $A = \alpha A_1 + (1-\alpha) A_2$

$$\Rightarrow I(X;Y) \geq \log\det\left(I + Q(\alpha A_1 + (1-\alpha)A_2)^{-1}\right)$$

(1+2) $\Rightarrow$ convexity of $A \mapsto \log\det(I + QA^{-1})$

$Q = B^*B \Rightarrow$ convexity of $A \mapsto \log\det(I + BA^{-1}B^*)$ #

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Minkowski's inequality

Def: for X a continuous complex random $\overbrace{\text{vector}}^{\text{of dimension } n}$, we set

$$N(X) := \frac{1}{\pi e} \exp\left(\frac{1}{n} h(X)\right).$$

Entropy power inequality: if X and Y are independent,

then $N(X+Y) \geq N(X) + N(Y)$.

So if $X \sim N_{\mathbb{C}}(0, A)$ and $Y \sim N_{\mathbb{C}}(0, B)$, then

$$N(X) = \frac{1}{\pi e} \exp\left(\frac{1}{n} \log\det(\pi e A)\right) = (\det A)^{1/n}$$

$$\Rightarrow \left(\det(\underbrace{A+B}_{=\mathrm{Cov}(X+Y)})\right)^{1/n} \geq (\det A)^{1/n} + (\det B)^{1/n}$$

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Lieb's inequality

Let X, Y, Z be three independent random vectors

$Y \longrightarrow Y+X \longrightarrow Y+X+Z$ forms a Markov chain,

so by the data processing inequality:

$$I(Y; X+Y+Z) \leq I(Y; X+Y)$$

i.e. $h(X+Y+Z) - \underbrace{h(X+Y+Z \mid Y)}_{=h(X+Z)} \leq h(X+Y) - \underbrace{h(X+Y \mid Y)}_{=h(X)}$

i.e. $h(X+Y+Z) + h(X) \leq h(X+Y) + h(X+Z)$

Consider $X \sim N_{\mathbb{C}}(0, I)$, $Y \sim N_{\mathbb{C}}(0, A)$, $Z \sim N_{\mathbb{C}}(0, B)$

$$\Rightarrow \log\det(I+A+B) + \underset{(=\log\det(I))}{0} \leq \log\det(I+A) + \log\det(I+B)$$

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