Random matrix theory: lecture 16

Generalizations: 1°) non-square matrices

Let H be a $n \times m$ random matrix with i.i.d. entries such that $\mathbb{E}(h_{11}) = 0$, $\mathbb{E}(|h_{11}|^2) = 1$, and $W^{(n)} = \frac{1}{n} H H^*$.

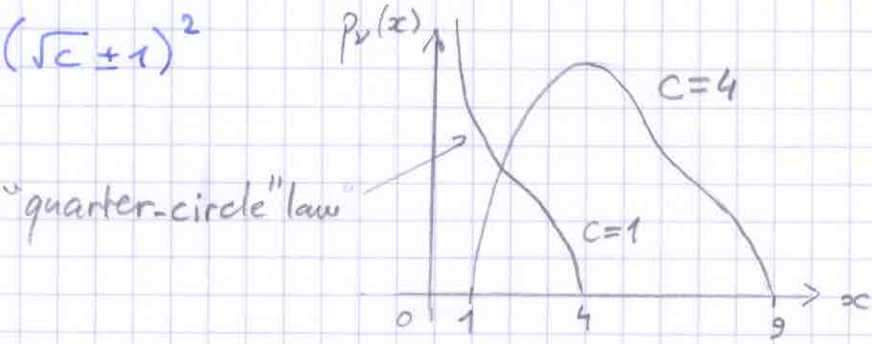
We assume that $m, n \to \infty$ with $\frac{m}{n} = c \geq 1$ fixed.

Let $\lambda_1^{(n)} \ldots \lambda_n^{(n)}$ be the eigenvalues of $W^{(n)}$. Then

$$F_n(t) := \frac{1}{n} \# \{ j : \lambda_j^{(n)} \leq t \} \xrightarrow[n \to \infty]{} \int_0^t p_\nu(x)\, dx \quad \text{a.s.} \quad \forall t \geq 0$$

where $p_\nu(x) = \frac{1}{2\pi} \sqrt{\left(\frac{x_+}{x} - 1\right)\left(1 - \frac{x_-}{x}\right)} \cdot 1_{x_- < x < x_+}$

and $x_\pm = (\sqrt{c} \pm 1)^2$



Remarks:

- the Stieltjes transform of p_ν, given by
$$g_\nu(z) = \int_{\mathbb{R}} \frac{1}{x - z} \cdot p_\nu(x)\, dx$$
is solution of the quadratic equation:
$$z\, g_\nu(z)^2 + (z + 1 - c)\, g_\nu(z) + 1 = 0$$
- if $\frac{m}{n} = c < 1$, then p_ν is the limiting distribution of the non-zero eigenvalues of $W^{(n)}$ (there are m of them: the remaining $n - m$ eigenvalues are equal to zero).

2°) insertion of a diagonal matrix Q

Let H be a $n \times n$ matrix with i.i.d. entries such that $\mathbb{E}(h_{11}) = 0$ and $\mathbb{E}(|h_{11}|^2) = 1$; let $Q^{(n)} = diag(q_1 .. q_n)$ (*) be a deterministic diagonal matrix and $W^{(n)} = \frac{1}{n} H Q^{(n)} H^*$, with corresponding eigenvalues $\lambda_1^{(n)} .. \lambda_n^{(n)}$.

(*) with $q_j \geq 0$

If $\frac{1}{n} \# \{1 \leq j \leq n : q_j \leq t\} \xrightarrow[n \to \infty]{} F_Q(t)$ $\forall t \geq 0$, with corresponding

× Stieltjes transform $g_Q(z) = \int_0^\infty \frac{1}{x-z} dF_Q(x)$, then

$$F_n(t) := \frac{1}{n} \# \{j : \lambda_j^{(n)} \leq t\} \xrightarrow[n \to \infty]{} F_W(t) \quad a.s. \quad \forall t \geq 0$$

whose Stieltjes transform $g_W(z)$ satisfies the equation:

$$z\, g_W(z)^2 + g_Q\left(-\frac{1}{g_W(z)}\right) = 0$$

<u>Example</u>: if $Q^{(n)} = I_n = diag(1, .., 1)$, then $g_Q(z) = \frac{1}{1-z}$

so $g_Q\left(-\frac{1}{g_W(z)}\right) = \frac{1}{1 + \frac{1}{g_W(z)}} = \frac{g_W(z)}{1 + g_W(z)}$

$$\Rightarrow \quad z\, g_W(z)^2 + \frac{g_W(z)}{1+g_W(z)} = 0$$

$$z\, g_W(z) (1 + g_W(z)) + 1 = 0$$

$$z\, g_W(z)^2 + z\, g_W(z) + 1 = 0$$

again, the "quarter-circle" law.

(non-negative definite)

3°) application: multiplication of random matrices

- It has been shown in (Silverstein, 1995) that if we replace the diagonal matrix $Q^{(n)}$ with any (deterministic) non-negative definite matrix $Q^{(n)}$ whose eigenvalues $q_1^{(n)} \dots q_n^{(n)}$ satisfy the same hypothesis as above, then the same conclusion holds: $z\, g_W(z)^2 + g_Q\left(-\frac{1}{g_W(z)}\right) = 0 \quad (*)$
- A further generalization of this result is that if $Q^{(n)}$ is a non-negative definite random matrix independent of H with limiting eigenvalue distribution F_Q and corresponding Stieltjes transform g_Q, then the result continues to hold!

Example: let $Q^{(n)} = \frac{1}{n} \tilde{H} \tilde{H}^*$, where $\tilde{H}$ and H are iid

Then we know that in the limit $n \to \infty$,

$$z\, g_Q(z)^2 + z\, g_Q(z) + 1 = 0$$

$$\xRightarrow{z \mapsto -\frac{1}{g_W(z)}} -\frac{1}{g_W(z)}\, g_Q\left(-\frac{1}{g_W(z)}\right)^2 - \frac{1}{g_W(z)}\, g_Q\left(-\frac{1}{g_W(z)}\right) + 1 = 0$$

$$\xRightarrow{\text{use } (*)} -z^2 g_W(z)^3 + z\, g_W(z) + 1 = 0$$

Cubic equation for g_W (NB: $W^{(n)} = \frac{1}{n^2} H \tilde{H} \tilde{H}^* H^*$)

4°) addition of random matrices

Let H be a $n \times n$ random matrix with i.i.d. entries such that $\mathbb{E}(h_{11}) = 0$ and $\mathbb{E}(|h_{11}|^2) = 1$; let $A^{(n)}$ be a deterministic $n \times n$ Hermitian matrix with eigenvalues $a_1^{(n)} \ldots a_n^{(n)}$ satisfying $\frac{1}{n} \#\{j : a_j^{(n)} \leq t\} \underset{n \to \infty}{\to} F_A(t)$ with corresponding Stieltjes transform $g_A(z)$; and let $W^{(n)} = A^{(n)} + \frac{1}{n} HH^*$ with eigenvalues $\lambda_1^{(n)} \ldots \lambda_n^{(n)}$.

Then

$$F_n(t) := \frac{1}{n} \#\{j : \lambda_j^{(n)} \leq t\} \underset{n \to \infty}{\longrightarrow} F_W(t) \quad \text{a.s. } \forall t \in \mathbb{R}$$

whose Stieltjes transform $g_W(z)$ satisfies the equation:

$$g_W(z) = g_A\left(z - \frac{1}{1 + g_W(z)} \right)$$

Remarks:

- The result again naturally generalizes to the case where $A^{(n)}$ is a random matrix independent of H.
- When $A^{(n)} = 0$, $g_A(z) = -\frac{1}{z}$ and we again recover the "quarter-circle" law.

5°) <u>all combinations of these are possible</u>!

The most general result is actually already contained in Marčenko-Pastur (1967) and Bai-Silverstein (1995). For reference, it says that:

- if $A^{(n)}$ is Hermitian $n \times n$ with limiting S.T. $g_A(z)$
- if $Q^{(n)}$ is deterministic diagonal $m \times m$ with limiting S.T. $g_Q(z)$
- if H is $n \times m$ with iid entries such that $\mathbb{E}(h_{11}) = 0$ (*), $\mathbb{E}(|h_{11}|^2) = 1$ and $\frac{m}{n} = c$
- then $W^{(n)} = A^{(n)} + \frac{1}{n} H Q^{(n)} H^*$ has a limiting eigenvalue distribution whose Stieltjes transform $g_W(z)$ is solution of the equation:

$$(!) \quad g_W(z) = g_A\left(z - \frac{c}{g_W(z)}\left(1 - \frac{1}{g_W(z)} \cdot g_Q\left(-\frac{1}{g_W(z)}\right)\right)\right)$$

The core of the proof is the one developed in the last lecture; we will therefore not redo it.

(*) This assumption can actually be dropped; we will come back to this later in the class.

Back to the addition of random matrices and an alternate proof of Wigner's theorem

(using Stieltjes transform)

- Let H be a $n \times n$ real symmetric matrix such that $\{h_{jk}, j \leq k\}$ are iid random variables with $\mathbb{E}(h_{11}) = 0$ and $\mathbb{E}(h_{11}^2) = 1$.
- Let $A^{(n)} = \operatorname{diag}(a_1 \ldots a_n)$ be deterministic ($a_j \in \mathbb{R}$) and such that $\frac{1}{n} \# \{1 \leq j \leq n : a_j \leq t\} \xrightarrow[n\to\infty]{} F_A(t)$ $\forall t \in \mathbb{R}$, with corresponding Stieltjes transform $g_A(z) = \int_{\mathbb{R}} \frac{1}{x-z} \, dF_A(x)$.
- Let $B^{(n)} = A^{(n)} + \frac{1}{\sqrt{n}} H$, with eigenvalues $\lambda_1^{(n)} \ldots \lambda_n^{(n)}$.

Theorem (Pastur et al.)

$$F_n(t) := \frac{1}{n} \# \{j : \lambda_j^{(n)} \leq t\} \xrightarrow[n\to\infty]{} F_B(t) \quad \text{a.s.} \quad \forall t \geq 0$$

where the Stieltjes transform of F_B satisfies the equation

$$g_B(z) = g_A(z + g_B(z)), \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Remarks:

- When $A^{(n)} = 0$, $g_A(z) = -\frac{1}{z}$, so $-g_B(z)(z + g_B(z)) = 1$,

$$g_B(z)^2 + z g_B(z) + 1 = 0, \quad g_B(z) = -\frac{z}{2} \pm \sqrt{\frac{z^2}{4} - 1}$$

and $p_B(x) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \operatorname{Im} g_B(x + i\varepsilon) = \frac{1}{2\pi} \sqrt{4 - x^2} \cdot 1_{|x| \leq 2}$

i.e. we recover Wigner's semi-circle law.

- The result generalizes in various directions:

  1°) complex Hermitian case

  2°) $A^{(n)}$ Hermitian with eigenvalues $a_1^{(n)} \dots a_n^{(n)}$

  3°) $A^{(n)}$ random & independent of H

<u>Proof of the result</u> (main ideas)

- Let M be a $n \times n$ real symmetric matrix and $z \in \mathbb{C} \setminus \mathbb{R}$. Then (matrix inversion lemma)

$$(M - zI_n)^{-1}_{11} = 1 / \left(m_{11} - z - m_1^T (M_1 - zI_{n-1})^{-1} m_1 \right)$$

$$\text{where} \quad \begin{array}{c} \\ M = \end{array} \begin{array}{l} \\ \left(\begin{array}{c|c} m_{11} & m_1^T \\ \hline m_1 & M_1 \\ & \end{array} \right) \\ \uparrow \\ \text{one column} \end{array} \begin{array}{l} \leftarrow \text{one row} \\ . \\ \\ \\ \end{array}$$

matrix with column & row k removed

Similarly,

$$\mathrm{Tr}\left((M - zI_n)^{-1}\right) = \sum_{k=1}^{n} 1 / \left(\underbrace{m_{kk}}_{k^{\text{th}}\text{ diag. coeff.}} - z - m_k^T (M_k - zI_{n-1})^{-1} \underbrace{m_k}_{k^{\text{th}}\text{ column without diag. coeff.}} \right)$$

- $G^{(n)}(z) := (B^{(n)} - zI_n)^{-1}$, $\quad G_k^{(n)}(z) := (B_k^{(n)} - zI_{n-1})^{-1}$

  $g_n(z) = \frac{1}{n} \mathrm{Tr}\, G^{(n)}(z)$

  Using the matrix inversion lemma, we obtain

$$g_n(z) = \frac{1}{n} \sum_{k=1}^{n} 1 / \left(b_{kk}^{(n)} - z - (b_k^{(n)})^T G_k^{(n)}(z)\, b_k^{(n)} \right)$$

(cf. formula last time)

• Now: $\begin{cases} b_{kk}^{(n)} = a_k + \frac{1}{\sqrt{n}} h_{kk} \simeq a_k \text{ as } n\to\infty \\ b_k^{(n)} = \frac{1}{\sqrt{n}} h_k \text{ (since this vector contains no diag. element)} \end{cases}$

so $g_n(z) \underset{n\to\infty}{\simeq} \frac{1}{n}\sum_{k=1}^{n} 1/\left(a_k - z - \frac{1}{n} h_k^T G_k^{(n)}(z) h_k\right)$

• Same reasoning as last time:

1°) $\frac{1}{n} h_k^T G_k^{(n)}(z) h_k \simeq \frac{1}{n} \operatorname{Tr} G_k^{(n)}(z)$ (w.h.p.)

2°) $\frac{1}{n} \operatorname{Tr} G_k^{(n)}(z) \simeq \frac{1}{n} \operatorname{Tr} G^{(n)}(z) = g_n(z)$

• Therefore: $g_n(z) \simeq \frac{1}{n}\sum_{k=1}^{n} 1/(a_k - z - g_n(z))$

(assumption on $A^{(n)}$ →) $\cong \int_{\mathbb{R}} \frac{1}{x - z - g_n(z)}\, dF_A(x)$

(definition →) $= g_A(z + g_n(z))$

so as $n\to\infty$, $g_n(z) \to g_B(z)$ solution of $g_B(z) = g_A(z + g_B(z))$

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Example: let $A^{(n)} = \begin{pmatrix} 0 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 0 \end{pmatrix}$; $A^{(n)}$ has the following limiting

pdf for its eigenvalues: $p_A(x) = \frac{1}{\pi\sqrt{4-x^2}} 1_{|x|<2}$

(cf. class on Toeplitz matrices)

The corresponding Stieltjes transform is $g_A(z) = \frac{1}{\sqrt{z^2-4}}$

$\Rightarrow g_B(z) = 1/\sqrt{(z+g_B(z))^2 - 4}$

i.e. $g_B(z)^4 + 2z\, g_B(z)^3 + (z^2-4)\, g_B(z)^2 - 1 = 0$

(quartic equation)