Random matrix theory : lecture 11

Distributions without random variables!

Def: a (probability) distribution on $\mathbb{R}$ is an application

$$\mu : \mathcal{B}(\mathbb{R}) \longrightarrow [0,1] \text{ such that:}$$

[$:=$ { Borel subsets of $\mathbb{R}$ } $:=$ smallest σ-field containing all open subsets of $\mathbb{R}$]

$$\begin{cases} \cdot\ \mu(\phi) = \phi, \quad \mu(\mathbb{R}) = 1 \\ \cdot\ \text{if } B_n \cap B_m = \phi \ \forall n \neq m, \text{ then } \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) \end{cases}$$

Remark: a distribution might therefore exist independently of any underlying random variable!

Def: the cumulative distribution function (cdf) associated to a distribution μ is the application $F_\mu : \mathbb{R} \to [0,1]$ defined as $F_\mu(t) := \mu(]-\infty, t])$, $t \in \mathbb{R}$.

Properties:

- $\lim_{t\to\infty} F_\mu(t) = \mu(\mathbb{R}) = 1$, $\lim_{t \to -\infty} F_\mu(t) = \mu(\phi) = 0$
- F_μ is non-decreasing: $t_1 \leq t_2 \Rightarrow F_\mu(t_1) \leq F_\mu(t_2)$
- F_μ is right-continuous: $\lim_{\varepsilon \downarrow 0} F_\mu(t+\varepsilon) = F_\mu(t) \quad \forall t \in \mathbb{R}$
- the knowledge of F_μ characterizes μ entirely! (and reciprocally, of course)

Two canonical classes of distributions

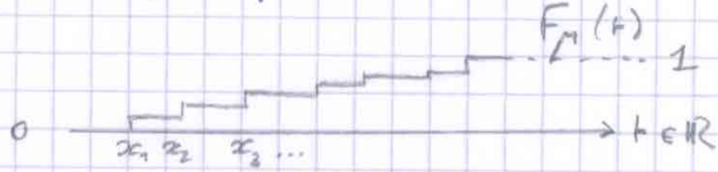
A) discrete distributions:

$\exists\ D = \{x_n\}_{n=1}^{\infty}$ (countable subset) such that $\mu(D) = 1$

(i.e. all the weight of the distribution μ is on D)

Let $p_n = \mu(\{x_n\})$: $\mu(B) = \sum_{x_n \in B} p_n$, $B \in \mathcal{B}(\mathbb{R})$

and $F_\mu(t) = \sum_{x_n \leq t} p_n$ step function



B) continuous distributions:

$\mu(B) = 0$ if $|B| = 0$ (in particular: $\mu(\{x\}) = 0 \ \forall x \in \mathbb{R}$)

("length" of B)

$\Rightarrow$ There exists a function p_μ such that $p_\mu(x) \geq 0$,

$\int_{\mathbb{R}} p_\mu(x)\,dx = 1$ and $\mu(B) = \int_B p_\mu(x)\,dx$, $B \in \mathcal{B}(\mathbb{R})$

p_μ is the probability density function (pdf) of μ

Moreover, $F_\mu(t) = \int_{-\infty}^{t} p_\mu(x)\,dx$ smooth function

NB: $p_\mu(x) = F_\mu'(x) = \frac{d\mu}{dx}(x)$

Riemann-Stieltjes integral with respect to a distribution μ

Def: a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if $\forall a<b$,

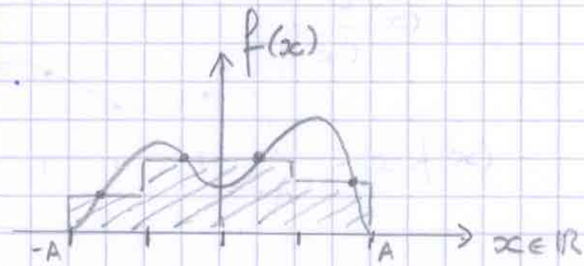
$$\{x \in \mathbb{R} : a < f(x) < b\} \text{ is an open subset of } \mathbb{R}$$

Let f be a continuous function on $\mathbb{R}$ such that $f(x)=0$ $\forall |x| > A$ (fixed) and μ be a (general) distribution on $\mathbb{R}$.

Let $-A = a_0 < a_1 < \ldots < a_n = A$ be a subdivision of $[-A, A]$.

× and $I_n := \sum_{j=1}^{n} f(\xi_j)\, \mu(]a_{j-1}, a_j])$.

where ξ_j is any point in $]a_{j-1}, a_j]$.



Thm:

For any continuous function f vanishing outside $[-A, A]$ and any sequence of subdivisions such that $\max_{1 \le j \le n} |a_j - a_{j-1}| \xrightarrow[n\to\infty]{} 0$ the sequence I_n converges to $I := \int_{\mathbb{R}} f(x)\, d\mu(x)$ as $n \to \infty$.

Alternate notations: • $I = \int_{\mathbb{R}} f(x)\, \mu(dx)$

• since $\mu(]a_{j-1}, a_j]) = F_\mu(a_j) - F_\mu(a_{j-1})$, one still writes

$$I = \int_{\mathbb{R}} f(x)\, dF_\mu(x) \quad \text{or} \quad I = \int_{\mathbb{R}} f(x)\, F_\mu(dx)$$

Remark:

The Riemann-Stieltjes integral can be extended to

× non-vanishing continuous functions on $\mathbb{R}$ (letting $A \to \infty$).

23.4.07

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Lebesgue's integral with respect to a distribution μ

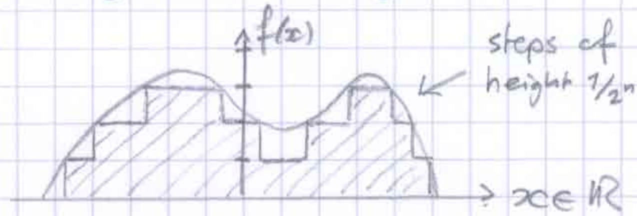
Def: a function $f : \mathbb{R} \to \mathbb{R}$ is Borel-measurable if $\forall a < b$

$$\{x \in \mathbb{R} : a < f(x) < b\} \text{ is a Borel subset of } \mathbb{R}$$

NB: this is a much weaker condition than being continuous!

- Let f be a non-negative Borel-measurable function on $\mathbb{R}$ (*)

and $I_n := \sum_{j=1}^{\infty} \frac{j-1}{2^n} \cdot \mu\left(\left\{x \in \mathbb{R} : \frac{j-1}{2^n} \leq f(x) < \frac{j}{2^n}\right\}\right)$



Remarks: • for fixed n, $I_n \in [0, \infty]$

• $I_n \leq I_{n+1} \quad \forall n$

(since the height of the steps is divided by 2 from n to $n+1$)

So $\lim_{n \to \infty} I_n = I$ exists and belongs to $[0, \infty]$.

I is the Lebesgue integral of f with respect to μ

and is denoted as $\int_{\mathbb{R}} f(x)\, d\mu(x)$. [same notation as Riemann-Stieltjes integral]

- Let f be a Borel-measurable function on $\mathbb{R}$ such that $\int_{\mathbb{R}} |f(x)|\, d\mu(x) < \infty$. Then

$$\int_{\mathbb{R}} f(x)\, d\mu(x) := \int_{\mathbb{R}} f^+(x)\, d\mu(x) - \int_{\mathbb{R}} f^-(x)\, d\mu(x)$$

where $f^+(x) := \max(0, f(x)) \geq 0$ and $f^-(x) := \max(0, -f(x)) \geq 0$.

(*) and μ be a (general) distribution on $\mathbb{R}$

Remarks:

- Both Riemann-Stieltjes and Lebesgue's integrals are well defined for $f: \mathbb{R} \to \mathbb{R}$ bounded and continuous (and they coincide for such f). Let us check this for Lebesgue: $\int_{\mathbb{R}} |f(x)|\, d\mu(x) \le \underbrace{\sup_{x \in \mathbb{R}} |f(x)|}_{< \infty} \cdot \underbrace{\int_{\mathbb{R}} 1 \cdot d\mu(x)}_{= \mu(\mathbb{R}) = 1} < \infty$.
- Lebesgue's integral is more general (except for some particular cases), so we will always refer implicitly to this second definition.

Special cases of distributions μ:

A) Integral with respect to a discrete distribution μ:

$$\int_{\mathbb{R}} f(x)\, d\mu(x) = \sum_{n=1}^{\infty} f(x_n)\, p_n \qquad p_n = \mu(\{x_n\})$$

B) Integral with respect to a continuous distribution μ:

$$\int_{\mathbb{R}} f(x)\, d\mu(x) = \int_{\mathbb{R}} f(x)\, p_\mu(x)\, dx \qquad p_\mu = \frac{d\mu}{dx}$$

Special cases of functions f:

1) For a given $t \in \mathbb{R}$, let $f(x) = 1_{]-\infty, t]}(x)$.
   f is Borel-measurable and bounded and
   $\int_{\mathbb{R}} 1_{]-\infty,t]}(x)\, d\mu(x) = \mu(]-\infty, t]) = F_\mu(t)$, cdf of μ.

2) For a given $t \in \mathbb{R}$, let $f(x) = e^{itx}$; f is bounded and continuous and $\int_{\mathbb{R}} e^{itx}\, d\mu(x) = \phi_\mu(t)$,
   Fourier transform or characteristic function of μ.

3) For a given $k \geq 0$, let $f(x) = x^k$; f is continuous and unbounded; $\int_{\mathbb{R}} |f(x)|\, d\mu(x)$ is therefore not necessarily finite. When this is the case, we define
   $\int_{\mathbb{R}} x^k\, d\mu(x) = m_k$ moment of order k of μ.

4) For a given $z \in \mathbb{C} \setminus \mathbb{R}$, let $f(x) = \frac{1}{x-z}$; f is complex-valued, bounded and continuous (since $z \notin \mathbb{R}$)
   $\int_{\mathbb{R}} \frac{1}{x-z}\, d\mu(x) = g_\mu(z)$ Stieltjes transform of μ

Weak convergence of sequences of distributions

Def: A sequence of distributions $(\mu_n)_{n=1}^{\infty}$ converges weakly to a distribution μ if

$$\lim_{n\to\infty} F_{\mu_n}(t) = F_\mu(t)$$

$\forall t \in \mathbb{R}$ continuity point of F_μ

(Unfortunate) notation: $\mu_n \underset{n\to\infty}{\Longrightarrow} \mu$

Two equivalent definitions:

Def': $\mu_n \underset{n\to\infty}{\Longrightarrow} \mu$ iff $\lim_{n\to\infty} \mu_n(]a,b]) = \mu(]a,b])$

$\forall a<b$ such that $\mu(\{a\}) = \mu(\{b\}) = 0$

(Proof: use $\mu(]a,b]) = F_\mu(b) - F_\mu(a)$)

Def'': $\mu_n \underset{n\to\infty}{\Longrightarrow} \mu$ iff $\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\, d\mu_n(x) = \int f(x)\, d\mu(x)$

$\forall f: \mathbb{R} \to \mathbb{R}$ bounded and continuous

(Proof: approximate $1_{]-\infty,t]}$ by a sequence of bdd and continuous functions; reciprocally, approximate f by a sequence of step functions)

Weak convergence and Fourier transform

Proposition (inversion formula)

The knowledge of the function $\phi_\mu(t) = \int_{\mathbb{R}} e^{itx}\, d\mu(x)$, $t \in \mathbb{R}$ characterizes μ entirely. Moreover, $\forall a<b$,

$$\mu(]a,b[) + \frac{1}{2}\mu(\{a,b\}) = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \cdot \phi_\mu(t)\, dt.$$

If μ is a continuous distribution with pdf p_μ, then

$$p_\mu(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}\, \phi_\mu(t)\, dt.$$

Proposition:

$$\mu_n \underset{n\to\infty}{\Rightarrow} \mu \quad \text{iff} \quad \lim_{n\to\infty} \phi_{\mu_n}(t) = \phi_\mu(t) \quad \forall t \in \mathbb{R}$$

Remarks:

- this proposition is of most importance in probability (it is used for proving the central limit theorem, e.g.)
- unfortunately, it is mostly useless for random matrices (explanation coming)