## Homework 7: Stieltjes transform

**Due date:** May 22 (Thursday).

**1. Basic properties:** Let $F$ be a probability distribution on $\mathbb{R}$ and $g_F : \mathbb{C}\backslash\mathbb{R} \to \mathbb{C}$ be its Stieltjes transform, defined as

$$g_F(z) = \int_{\mathbb{R}} \frac{1}{x-z}\, dF(x), \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

a) Writing $z = u + iv$, decompose $g_F(z)$ into real and imaginary parts.

b) It can be shown that $g_F$ is analytic on $\mathbb{C}\backslash\mathbb{R}$ (no proof required here).

c) Show that $\operatorname{Im} g_F(z) > 0$, if $\operatorname{Im} z > 0$.

d) Show that $\lim_{v\to+\infty} v\,|g_F(iv)| = 1$.

e) Show that $g_F(\overline{z}) = \overline{g_F(z)}$.

NB: If a function $g$ satisfies properties b) to e), then it is the Stieltjes transform of a distribution $F$ (BTW, property c) ensures that $F$ is non-decreasing and property d) ensures that $F(+\infty) - F(-\infty) = 1$).

**2. Inversion formula:** a) Let $F$ be any probability distribution on $\mathbb{R}$. For any $a < b$ continuity points of $F$, prove that

$$\frac{1}{\pi} \lim_{\varepsilon\downarrow 0} \int_a^b \operatorname{Im}\, g_F(x + i\varepsilon)\, dx = F(b) - F(a).$$

b) Assume now that $F$ has a pdf $p_F$. Show that for any $x \in \mathbb{R}$,

$$p_F(x) = \frac{1}{\pi} \lim_{\varepsilon\downarrow 0} \operatorname{Im} g_F(x + i\varepsilon).$$

**3.** a) Let $x_0 \in \mathbb{R}$, $y_0 \geq 0$ and $g_0(z) = \frac{1}{x_0 - iy_0 - z}$ for $\operatorname{Im} z > 0$. Deduce the distribution corresponding to the Stieltjes transform $g_0$ (separate the cases $y_0 = 0$ and $y_0 > 0$). What are the moments of this distribution?

b) Let $g_F$ be the Stieltjes transform solution of the quadratic equation

$$g_F(z)^2 + z\, g_F(z) + 1 = 0.$$

Deduce what distribution $F$ corresponds to $g_F$.

c) Let $g_F$ be the Stieltjes transform solution of the quadratic equation

$$z\, g_F(z)^2 + z\, g_F(z) + 1 = 0.$$

Deduce what distribution $F$ corresponds to $g_F$.

NB: Both the above quadratic equations have two solutions, but notice that for each of them, only one solution is a Stieltjes transform satisfying the properties of Exercise 1.

please turn the page.

**4. Matrix inversion lemma**

Let $H$ be a $n \times n$ real symmetric matrix and $z \in \mathbb{C} \backslash \mathbb{R}$. Show that

$$((H - zI_n)^{-1})_{11} = \frac{1}{h_{11} - z - h_1^T \, (H_1 - zI_{n-1})^{-1} \, h_1},$$

where $h_1$ is the first column of $H$ without its first element and $H_1$ is the matrix $H$ with first row and first column removed.