Homework 5

Due date: April 21 (Monday).

1. In the class, we have seen that if $m \geq n$ and $W = HH^*$ with H an $n \times m$ random matrix with i.i.d.$\sim \mathcal{N}_{\mathbb{C}}(0,1)$ entries, then the distribution of W is given by

$$P_W(W) = C_{n,m} \exp(-\mathrm{Tr}(W)) (\det W)^{m-n}.$$

The constant in front is actually given by

$$C_{n,m} = \frac{1}{\pi^{n(n-1)/2} \prod_{j=1}^n \Gamma(m-j+1)},$$

where $\Gamma(x)$ is the Euler Gamma function defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt.$$

We know that $\Gamma(1) = 1$ and $\Gamma(x+1) = x\,\Gamma(x)$ for $x > 0$ (you may want to prove this, as a "starter"). From this, we deduce that $\Gamma(k) = (k-1)!$ for every integer $k \geq 1$.

a) Using the above (and *not* the expression found in the class for the joint eigenvalue distribution), compute $\mathbb{E}(\det W)$.

b) For which values of $m \geq n$ is $\mathbb{E}(1/\det W)$ finite? Compute this expression.

c) Let $\psi(x)$ be the Euler digamma function defined as

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = \frac{d}{dx}(\log \Gamma(x)).$$

Prove that $\psi(x+1) = \psi(x) + \frac{1}{x}$ for $x > 0$. From this, we deduce that

$$\psi(k) = \sum_{j=1}^{k-1} \frac{1}{j} + \psi(1)$$

for every integer $k \geq 1$. It turns out also that

$$\psi(1) = \int_0^\infty \log(r) \, e^{-r} \, dr = -\gamma,$$

where $\gamma = 0.57721...$ is the Euler constant.

d) Compute $\mathbb{E}(\log \det W)$.

please turn the page.

2. Recall if Q is an $n \times n$ positive-definite matrix, then

$$\int_{\mathbb{C}^n} dz \, \frac{1}{\pi^n \det Q} \exp\left(-z^* Q^{-1} z\right) = 1.$$

a) Let H be an $n \times n$ matrix with i.i.d.$\sim \mathcal{N}_{\mathbb{C}}(0,1)$ entries. Use the above fact to prove that

$$\mathbb{E}_H \left(\frac{1}{\det(I + HH^*)} \right) = \mathbb{E}_z \left(\frac{1}{(1 + \|z\|^2)^n} \right),$$

where $z \sim \mathcal{N}_{\mathbb{C}}(0, I_n)$ (n-variate vector).

b) Using the same technique, prove the following more general statement:

If H is an $n \times m$ matrix with i.i.d.$\sim \mathcal{N}_{\mathbb{C}}(0,1)$ entries, then

$$\mathbb{E}_H \left(\frac{1}{\det(I + HH^*)^k} \right) = \mathbb{E}_Z \left(\frac{1}{\det(I + ZZ^*)^m} \right),$$

where Z is an $n \times k$ matrix with i.i.d.$\sim \mathcal{N}_{\mathbb{C}}(0,1)$ entries.