Homework 4

Due date: April 7 (Monday).

1. Joint singular value distribution
Based on the results of the class, compute the joint singular value distribution of H in the following three cases:

a) H is a $n \times n$ real matrix picked from the GOE.

b) H is a $n \times m$ real matrix with i.i.d.$\sim \mathcal{N}_{\mathbb{R}}(0,1)$ entries (we assume $m \geq n$).

c) H is a $n \times n$ real matrix picked from the COE.

Hint: Among these three computations, one is straightforward, one is quite easy, and one is probably quite difficult, if not infeasible! The goal here is at least to identify these three cases.

2. Computation of the marginal eigenvalue distributions of the CUE
Let U be a $n \times n$ complex unitary matrix picked according to the Haar distribution on $U(n)$. U has n complex eigenvalues $\lambda_1, ..., \lambda_n$ which are all located on the unit circle: $|\lambda_j| = 1$. We may therefore write $\lambda_j = e^{i\theta_j}$, where $\theta_j \in [0, 2\pi[$. The joint distribution of (the arguments of) the eigenvalues of U is given by

$$p(\theta_1, ..., \theta_n) = C_n \prod_{j<k} \left| e^{i\theta_k} - e^{i\theta_j} \right|^2, \quad \theta_j \in [0, 2\pi[.$$

The goal of this exercise is to show that the first order and second order marginal distributions of $p(\theta_1, ..., \theta_n)$ are given respectively by

$$p(\theta) = \frac{1}{2\pi} \quad \text{and} \quad p(\theta, \varphi) = \frac{1}{4\pi^2} \frac{n}{n-1} \left(1 - \left(\frac{\sin(n(\theta - \varphi)/2)}{n \sin((\theta - \varphi)/2)} \right)^2 \right), \quad \theta, \varphi \in [0, 2\pi[.$$

a) Following the reasoning developed in the class for the Complex Wishart Ensemble, express $p(\theta_1, ..., \theta_n)$ in terms of the self-reproducing kernel $K(\theta, \varphi) = \sum_{l=0}^{n-1} e^{il\theta} \, \overline{e^{il\varphi}}$.

b) Show that (0) $K(\varphi, \theta) = \overline{K(\theta, \varphi)}$.

(i) $\int_0^{2\pi} K(\theta, \theta) \, d\theta = 2\pi \, n$.

(ii) $\int_0^{2\pi} K(\theta, \varphi) \, K(\varphi, \psi) \, d\varphi = 2\pi \, K(\theta, \psi)$.

c) For $m \in \{1, \ldots, n\}$, let $D_m(\theta_1, \ldots, \theta_m) = \det \left(\{K(\theta_j, \theta_k)\}_{j,k=1}^m \right)$. Prove Mehta's lemma:

$$\int_0^{2\pi} D_m(\theta_1, \ldots, \theta_m) \, d\theta_m = 2\pi \, (n - m + 1) \, D_{m-1}(\theta_1, \ldots, \theta_{m-1}).$$

Hint: recall that for a $m \times m$ matrix A and for any $j \in \{1, \ldots, m\}$,

$$\det A = \sum_{k=1}^m (-1)^{j+k} \, a_{jk} \, \det(A(j,k)) = \sum_{k=1}^m (-1)^{j+k} \, a_{kj} \, \det(A(k,j)),$$

where $A(j,k)$ is the matrix A whose row j and column k have been removed.

d) Deduce the formulas stated above for $p(\theta)$ and $p(\theta, \varphi)$.