**Homework 3: Joint distribution of eigenvalues**

**Due date:** March 17 (Monday).

Let $a, b, c$ be three independent random variables such that $a, c \sim \mathcal{N}_{\mathbb{R}}(0,1)$ and $b \sim \mathcal{N}_{\mathbb{R}}(0,1/2)$, and let

$$1.\ H_1 = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad \text{and} \quad 2.\ H_2 = \begin{pmatrix} a & c \\ c & a \end{pmatrix}.$$

The goal of the exercise is to compute the joint eigenvalue distributions of both $H_1$ and $H_2$, following two different approaches:

A. use the "Jacobian method" described in the class.

B. compute directly the eigenvalues of the matrix and look for their joint distribution.

**Guidelines:**

A.1. Since $H_1$ is symmetric, its eigenvalues $\lambda, \mu$ are real and there exists $\theta \in [0, \frac{\pi}{2}]$ such that

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

- Compute the joint distribution $p(a, b, c)$ of the entries.
- Write explicitly the change of variables $a(\lambda, \mu, \theta)$, $b(\lambda, \mu, \theta)$, $c(\lambda, \mu, \theta)$ and compute its Jacobian

$$J(\lambda, \mu, \theta) = \det \left. \begin{pmatrix} \frac{\partial a}{\partial \lambda} & \frac{\partial a}{\partial \mu} & \frac{\partial a}{\partial \theta} \\ \frac{\partial b}{\partial \lambda} & \frac{\partial b}{\partial \mu} & \frac{\partial b}{\partial \theta} \\ \frac{\partial c}{\partial \lambda} & \frac{\partial c}{\partial \mu} & \frac{\partial c}{\partial \theta} \end{pmatrix} \right|_{(\lambda,\mu,\theta)}.$$

- Compute the joint distribution

$$p(\lambda, \mu, \theta) = p(a(\lambda, \mu, \theta), b(\lambda, \mu, \theta), c(\lambda, \mu, \theta))\ |J(\lambda, \mu, \theta)|$$

and deduce an expression for $p(\lambda, \mu)$.

- Compute also

$$\mathbb{E}(\lambda + \mu), \quad \mathbb{E}(\lambda\mu) \quad \text{and} \quad \mathbb{E}(\max\{\lambda, \mu\}).$$

Watch out that the first two computations are particularly easy!

- Compute finally the marginal distribution $p(\lambda) = \int_{\mathbb{R}} p(\lambda, \mu)\, d\mu$. How does it look?

A.2. Redo the same analysis as above, but watch out that the eigenvectors of $H_2$ are now deterministic!

please turn the page

B. For this part, you might need the following facts about **Gaussian random vectors**:

Two real Gaussian random variables $(x, y)$ are said to form a (2-variate) Gaussian random vector if for any $\alpha, \beta \in \mathbb{R}$, $\alpha x + \beta y$ is also a Gaussian random variable [in this definition, we adopt the convention that if a random variable is constant (for example equal to zero), then we say that it is a Gaussian random variable of variance zero].

For example, if $x$ and $y$ are independent Gaussian random variables, then $(x, y)$ forms a Gaussian random vector.

The joint distribution of a generic Gaussian random vector $(x, y)$ is entirely determined by the means $\bar{x} = \mathbb{E}(x)$, $\bar{y} = \mathbb{E}(y)$ and the covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_x^2 \end{pmatrix},$$

where $\sigma_x^2 = \mathbb{E}((x - \bar{x})^2)$, $\sigma_y^2 = \mathbb{E}((y - \bar{y})^2)$ and $\rho\sigma_x\sigma_y = \mathbb{E}((x - \bar{x})(y - \bar{y}))$.

In particular, if $\Sigma$ is non-singular, then

$$p(x, y) = \frac{1}{2\pi\sqrt{\det \Sigma}} \exp\left( -\frac{1}{2} \begin{pmatrix} x - \bar{x} \\ y - \bar{y} \end{pmatrix}^T \Sigma^{-1} \begin{pmatrix} x - \bar{x} \\ y - \bar{y} \end{pmatrix} \right),$$

and we deduce from this formula that if $\Sigma$ is diagonal, then $x$ and $y$ are independent.