Homework 2

Due date: March 10 (Monday).

1. Show that for any $n \times n$ positive definite matrix Q, the map $\Sigma \mapsto \log\det(I + Q\Sigma^{-1})$ is convex on the set of $n \times n$ positive definite matrices Σ, using the following information theoretic inequality:

$$I(X; X+Z) \geq \log\det(I + Q\Sigma^{-1}),$$

if X, Z are n-dimensional complex and jointly continuous random vectors such that $X \sim \mathcal{N}_{\mathbb{C}}(0, Q)$, Z has covariance matrix $\mathbb{E}(ZZ^*) = \Sigma$ and X, Z are independent.

2. Matrix norms

For a given $n \times n$ matrix $A = (a_{jk})$, we define two norms:

$$|||A|||_2 = \sup_{x \in \mathbb{C}^n : \|x\|=1} \|Ax\| \quad \text{and} \quad \|A\|_2^2 = \frac{1}{n}\mathrm{Tr}(AA^*).$$

a) *Relying only on these definitions*, show that

$$\left|\frac{1}{n}\mathrm{Tr}(A)\right| \leq \|A\|_2 \leq |||A|||_2.$$

b) For two $n \times n$ matrices A and B, show moreover that

$$|||AB|||_2 \leq |||A|||_2 \, |||B|||_2, \quad \|AB\|_2 \leq |||A|||_2 \, \|B\|_2$$

and find an example of matrices A, B for which

$$\|AB\|_2 > \|A\|_2 \, \|B\|_2.$$

c) Once you have shown the above statements, show that the following holds:

$$|||A|||_2 = \max_{j \in \{1,\ldots,n\}} \sigma_j, \quad \text{and} \quad \|A\|_2^2 = \frac{1}{n}\sum_{j=1}^n \sigma_j^2,$$

where $\sigma_1, \ldots, \sigma_n \geq 0$ are the singular values of A.

d) **Asymptotic equivalence:** let $(A^{(n)}, B^{(n)})_{n=1}^\infty$ be two sequences of $n \times n$ matrices such that

$$\lim_{n\to\infty} \|A^{(n)} - B^{(n)}\|_2 = 0.$$

Is this condition sufficient to ensure that for *any fixed* $m \geq 0$,

$$\lim_{n\to\infty} \left|\frac{1}{n}\mathrm{Tr}\left((A^{(n)})^m\right) - \frac{1}{n}\mathrm{Tr}\left((B^{(n)})^m\right)\right| = 0 \quad ?$$

If not, what additional condition(s) would be needed?

Hint: use the above statements together with the following fact:

$$(A^{(n)})^m - (B^{(n)})^m = \sum_{j=1}^m (B^{(n)})^{j-1}(A^{(n)} - B^{(n)})(A^{(n)})^{m-j}.$$