Principles of Digital Communications | EPFL Summer Semester 2007
April 26, 2007

Midterm

This is a closed-book exam, but a single page of hand-written notes is allowed. Calculators and cell-phones are not allowed.

You have 1 hour 45 minutes to solve the problems.

Try to do side calculations on a separate sheet and report well organized solutions. If we can't read it we can't grade it.

If you need additional pages, do not forget to write your name!

Good Luck!!

Name: ______________________________

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1. **Bhattacharyya Bound (13 points)**

When $Y \in \mathbb{R}$ is an continuous random variable, the Bhattacharyya bound states that

$$Pr\{Y \in \mathcal{B}_{i,j} | H = i\} \leq \sqrt{\frac{P_H(j)}{P_H(i)}} \int_{y \in \mathbb{R}} \sqrt{f_{Y|H}(y|i) f_{Y|H}(y|j)}\, dy,$$

where i, j are two possible hypotheses and $\mathcal{B}_{i,j} = \{y \in \mathbb{R} : P_H(i) f_{Y|H}(y|i) \leq P_H(j) f_{Y|H}(y|j)\}$. In this problem $\mathcal{H} = \{0, 1\}$ and $P_H(0) = P_H(1) = 0.5$.

(a) (2 points) Write a sentence that expresses the meaning of $Pr\{Y \in \mathcal{B}_{0,1} | H = 0\}$. (You may write in English or French. Use words that have operational meaning.)

(b) (2 points) Do the same but for $Pr\{Y \in \mathcal{B}_{0,1} | H = 1\}$. (Note that we have written $\mathcal{B}_{0,1}$ and not $\mathcal{B}_{1,0}$.)

(c) (3 points) Evaluate the right hand side of the Bhattacharyya bound for the special case $f_{Y|H}(y|0) = f_{Y|H}(y|1)$.

(d) (4 points) Evaluate the Bhattacharyya bound for the following (Laplacian noise) setting:

$$
\begin{aligned}
H = 0 : \qquad & Y = -a + Z \\
H = 1 : \qquad & Y = a + Z,
\end{aligned}
$$

where $a \in \mathbb{R}_+$ is a constant and $f_Z(z) = \frac{1}{2}\exp(-|z|)$, $z \in \mathbb{R}$. Hint: it does not matter if you evaluate the bound for $H = 0$ or $H = 1$.

(e) (2 points) For which value of a should the bound give the result obtained in (1c)? Verify that it does. Check your previous calculations if it does not.

2. **Irrelevance (5 points)**

Assume that H is a random variable that corresponds to a hypothesis. Let Y_s and Y_i be two more random variables, that are observations.

Definition: We say that $H \rightarrow Y_s \rightarrow Y_i$ forms a Markov chain if

$$f_{Y_i|H,Y_s}(y_i|i, y_s) = f_{Y_i|Y_s}(y_i|y_s)$$

for all possible values of y_i, i and y_s.

(a) (3 points) Show that $H \rightarrow Y_s \rightarrow Y_i$ forms a Markov chain **if and only if** $Y_i \rightarrow Y_s \rightarrow H$ forms a Markov chain, *i.e.*, if and only if

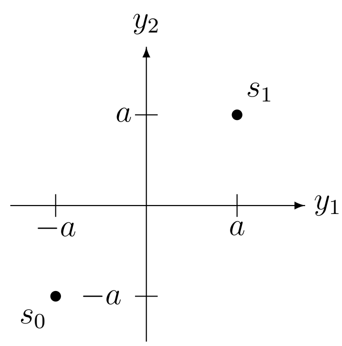
$$P_{H|Y_s,Y_i}(i|y_s, y_i) = P_{H|Y_s}(i|y_s)$$

holds for all values of i, y_s and y_i.

(b) (2 points) We know from the homework that if $f_{Y_i|H,Y_s}(y_i|i, y_s)$ does not depend on i (in other words, if $H \rightarrow Y_s \rightarrow Y_i$ forms a Markov chain), then Y_s is a sufficient statistic and Y_i is irrelevant. Using part (a), this tells us that if $P_{H|Y_s,Y_i}(i|y_s, y_i) = P_{H|Y_s}(i|y_s)$ for all values of i, y_s and y_i, then Y_s is a sufficient statistic (and Y_i is irrelevant).
Is this intuitive? Explain why / why not.

3. **Antipodal Signaling (6 points)**

Consider the following signal constellation:



Assume that $\boldsymbol{s}_1$ and $\boldsymbol{s}_0$ are used for communication over the Gaussian vector channel. More precisely:

$$H = 0: \quad \boldsymbol{Y} = \boldsymbol{s}_0 + \boldsymbol{Z},$$
$$H = 1: \quad \boldsymbol{Y} = \boldsymbol{s}_1 + \boldsymbol{Z},$$

where $\boldsymbol{Z} \sim \mathcal{N}(\boldsymbol{0}, \sigma^2 I_2)$. Hence, $\boldsymbol{Y}$ is a vector with two components $\boldsymbol{Y} = (Y_1, Y_2)$.

(a) (3 points) Argue why Y_1 is *not* a sufficient statistic.

(b) (3 points) Give a different signal constellation with two signals $\tilde{\boldsymbol{s}}_0$ and $\tilde{\boldsymbol{s}}_1$ such that, when using them in the above communication setting, Y_1 is a sufficient statistic.

4. **Hypothesis Testing (11 points)**

Consider a binary hypothesis testing problem in which the hypotheses $H = 0$ and $H = 1$ occur with probability $P_H(0)$ and $P_H(1) = 1 - P_H(0)$, respectively. The observation $\boldsymbol{Y}$ is a sequence of zeros and ones of length $2k$, where k is a fixed integer. When $H = 0$, each component of $\boldsymbol{Y}$ is 0 or a 1 with probability $\frac{1}{2}$ and components are independent. When $H = 1$, $\boldsymbol{Y}$ is chosen uniformly at random from the set of all sequences of length $2k$ that have an equal number of ones and zeros. There are $\binom{2k}{k}$ such sequences.

(a) (3 points) What is $P_{\boldsymbol{Y}|H}(\boldsymbol{y}|0)$? What is $P_{\boldsymbol{Y}|H}(\boldsymbol{y}|1)$?

(b) (3 points) Find a maximum likelihood decision rule. What is the single number you need to know about $\boldsymbol{y}$ to implement this decision rule?

(c) (2 points) Find a decision rule that minimizes the error probability.

(d) (3 points) Are there values of $P_H(0)$ and $P_H(1)$ such that the decision rule that minimizes the error probability always decides for only one of the alternatives? If yes, what are these values, and what is the decision?