Midterm

There are 9 pages. If you need any extra pages do not forget to mark your name! You have 1 hour 45 minutes. It is not necessarily expected that you finish all problems. Do not loose too much time on each problem but try to collect as many points as possible.

Closed-book, no calculators, cell-phones, crypt sheet or stolen glances to your neighbors are allowed. Write what is relevant to the question!

Good Luck!!

Name: ______________________________

Ex I	/ 25
Ex II	/ 15
Ex III	/ 15
Ex IV	/ 15
Total	/ 70

I. **Dice Tossing [25 points]**

You have two dices, one fair and one loaded (*truqué*). A friend told you that the loaded dice produces a 6 with probability $\frac{1}{4}$, and the other values with uniform probabilities. You do not know a priori which one is fair or which one is loaded. You pick with uniform probabilities one of the two dices, and perform N consecutive tosses (*lancés*) with the dice you have chosen. Let

$$Y = (Y_1, \cdots, Y_N)$$

be the sequence of numbers observed.

1. [10 points] Based on the sequence of observations Y, find the decision rule to determine whether the dice you have chosen is loaded. Your decision rule should maximize the probability of correct decision.

2. [5 points] Identify a compact sufficient statistic for this hypothesis testing problem, call it S. Justify your answer. [Hint: $S \in \mathbb{N}$.]

3. [5 points] Find the Bhattacharyya bound on the probability of error. [Hint: Use $f_{S|H}(s|i)$ instead of $f_{Y|H}(y|i)$. Recall $\sum_{i=1}^{N} \alpha = \frac{1-\alpha^{N+1}}{1-\alpha}$, for $|\alpha| < 1$.]

4. [5 points] Assume that, in addition to Y, you observe the count of even and odd numbers that appears during the N successive dice tosses. Denote them by N_e and N_o, respectively. Of course, $N = N_e + N_o$. Are these two numbers relevant to your hypothesis testing problem? Justify your answer.

II. **Who Wants to Be a Millionaire [15 points]**

Assume you are at a quiz show. You are shown three boxes which look identical from the outside, except they have labels 0, 1, and 2, respectively. Exactly one of them contains one million Swiss francs, the other two contain nothing. A computer randomly chooses a box with uniform probability. Let A be the random variable which denotes his choice, $A \in \{0, 1, 2\}$. The quizmaster now eliminates from the remaining two boxes one that does not contain the prize. This means that if neither of the two remaining boxes contain the prize then the quizmaster eliminates one with uniform probability. Otherwise, he simply eliminates the one which does not contain the prize. Let B denote the random variable corresponding to the box eliminated by the quizmaster, $B \in \{0, 1, 2\}$, and let C denote the remaining box. You are asked to choose one of the three boxes knowing A and B.

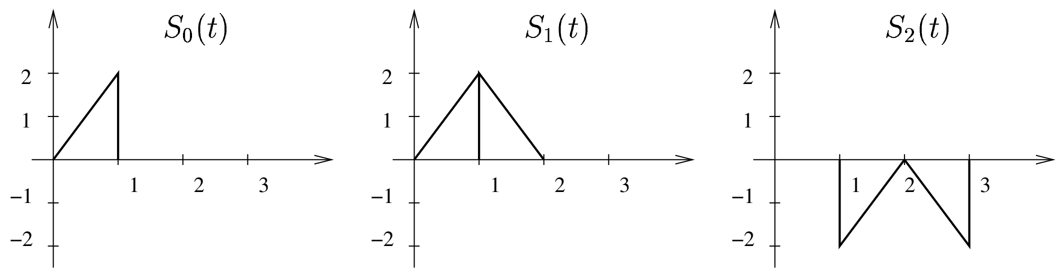
1. [5 points] Formulate this as a hypotheses testing problem. What is the set of hypotheses, what are the observations, and what are the priors?

2. [5 points] Write down the general rule for the optimal decision. Assume that $A = 0$ and $B = 1$. What is the optimal decision?

3. [5 points] What is the optimal decision in the general case?

III. **Signal Space Representation [15 points]**

Consider the following functions $S_0(t)$, $S_1(t)$ and $S_2(t)$.



1. [5 points] Using the Gram-Schmidt procedure, determine a basis of the space spanned by $\{s_0(t), s_1(t), s_2(t)\}$. Denote the basis functions by $\phi_0(t)$, $\phi_1(t)$ and $\phi_2(t)$.

2. [5 points] Let

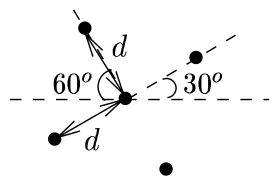
$$V_1 = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad V_2 = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

be two points in the space spanned by $\{\phi_0(t), \phi_1(t), \phi_2(t)\}$. What is their corresponding signal, $V_1(t)$ and $V_2(t)$? (You can simply draw a detailed graph.)

3. [5 points] Compute $\int V_1(t)V_2(t)dt$.

IV. **Signal Constellations under AWGN [15 points]**

Consider the following 5-ary hypothesis testing problems. We assume that, the prior is uniform, the signal constellation consists of elements of $\mathbb{R}^2$, and the noise is additive Gaussian with independent components of variance σ^2 in each dimension.



1. [2 points] Draw the boundaries of the decision regions into the above figure.
2. [3 points] Give the exact probability of error for the middle point of the constellation.

3. [5 points] Find an upper bound on the probability of error, Pr{e}, using the union bounding technique.

4. [5 points] Consider the following set of parameters:

$$\sigma^2 = 1, d = 1 \quad (1)$$
$$\sigma^2 = 2, d = 2 \quad (2)$$
$$\sigma^2 = 4, d = 2 \quad (3)$$

Which one do you prefer and why?