

Midterm

You have 1 hour 45 minutes. It is not necessarily expected that you finish all problems. Do not loose too much time on each problem but try to collect as many points as possible.

Closed-book, no calculators, cell-phones, crypt sheet or stolen glances to your neighbors are allowed. Write only what is relevant to the question!

Good Luck!!

Name: ______________________________

Ex I	/ 20
Ex II	/ 20
Ex III	/ 20
Ex IV	/ 20
Total	/ 80

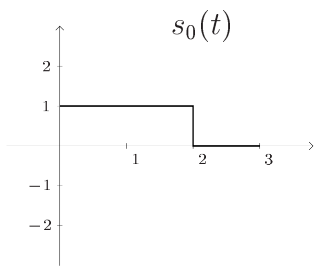
I. **Projection onto a Non-Orthogonal Set [20 points]**

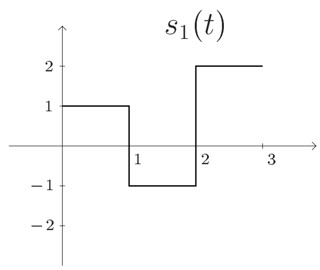
Consider the following ternary hypothesis testing problem, where $H \in \mathcal{H} = \{0, 1, 2\}$.

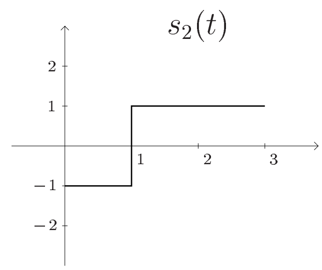
If $H = 0$ we transmit $s_0(t)$,

$H = 1$ we transmit $s_1(t)$,

$H = 2$ we transmit $s_2(t)$,





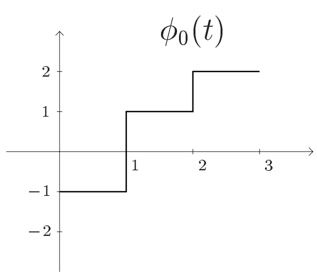


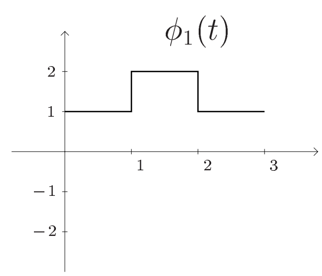
The received signal is

$$R(t) \quad = \quad s_i(t) + N(t),$$

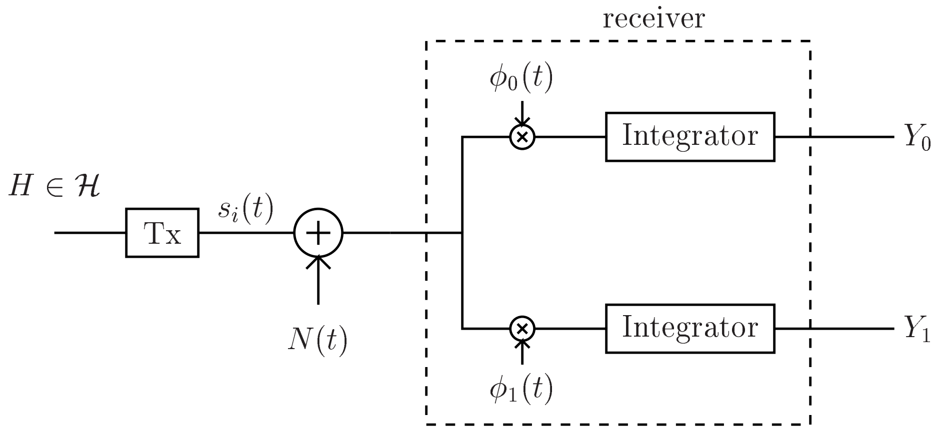
where $N(t)$ denotes a white Gaussian noise process with double-sided power spectral density equal to $\frac{N_0}{2}$.

1. [5 points] What is the dimension of $\mathbb{S} = \text{span}\{s_0(t), s_1(t), s_2(t)\}$? [Hint: It is not necessary to do a Gram-Schmidt decomposition to answer this question!]





2. [15 points] Consider the functions $\phi_0(t)$, $\phi_1(t)$ and assume the receiver applies the following processing:



The output of the receiver is the vector $Y = \begin{pmatrix} Y_0 \\ Y_1 \end{pmatrix}$.

For each hypothesis $i \in \{0, 1, 2\}$, write down the corresponding likelihood function $f_{Y|H}(\mathbf{y}|i)$.

II. **Data Dependent Noise [20 points]**

Consider the following binary Gaussian hypothesis testing problem with data dependent noise.

Under hypothesis H_0 the transmitted signal is $s_0 = -1$ and the received signal is $Y = s_0 + Z_0$, where Z_0 is zero-mean Gaussian with variance one.

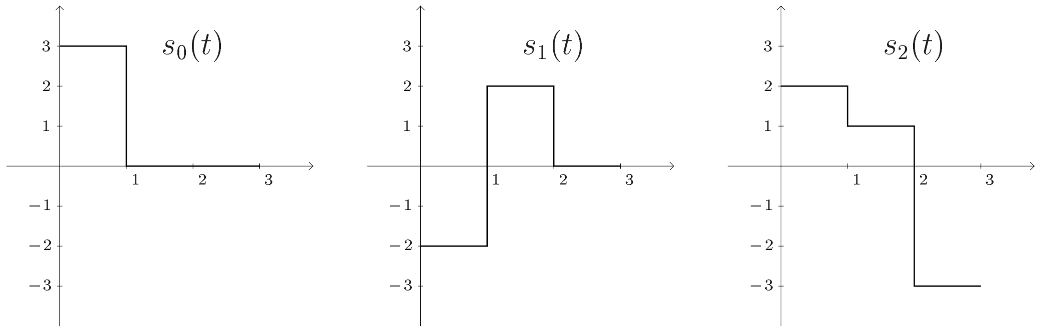
Under hypothesis H_1 on the other hand, the transmitted signal is $s_1 = 1$ and the received signal is $Y = s_1 + Z_1$, where Z_1 is zero-mean Gaussian with variance σ^2. Assume that the prior is uniform.

1. [5 points] Starting from first principles write down the optimal decision rule as a function of the parameter σ^2 and the received signal Y.
2. [10 points] For the value $\sigma^2 = \exp(4)$ compute the decision regions.
3. [5 points] Give as simple expressions as possible for the error probabilities $Pr\{e|0\}$ and $Pr\{e|1\}$.

Hint: It might be handy to recall that the solutions of the quadratic equation $ax^2 + bx + c = 0$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

III. **Gram-Schmidt and Minimum Energy [20 points]**

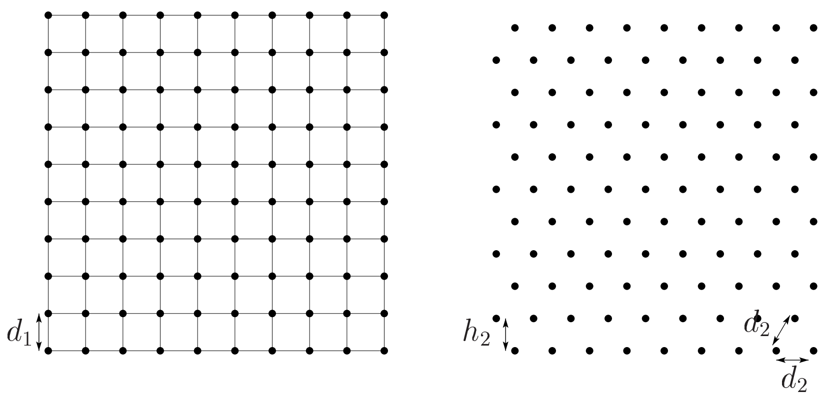
Consider the following signal constellation $\{s_0(t), s_1(t), s_2(t)\}$ where the signals are used with equal probability.



1. [5 points] Using the Gram-Schmidt procedure, determine a basis of the space spanned by $\{s_0(t), s_1(t), s_2(t)\}$.
2. [5 points] Compute the average energy per symbol of this constellation.
3. [5 points] Derive the equivalent minimum energy constellation by properly shifting the original constellation. Compute its average energy per symbol.
4. [5 points] Draw the corresponding minimum energy signals.

IV. **Comparison of Two Constellations [20 points]**

Consider the following two m-ary hypothesis testing problems where m is very large. In both cases the prior is uniform, the signal constellations are elements of $\mathbb{R}^2$ and the noise is additive Gaussian with independent components of variance σ^2 in each dimension. Signal constellation $\mathcal{S}_1$ consists of points of the two-dimensional integer grid. The signal constellation $\mathcal{S}_2$ is the so-called "hexagonal" constellation (which you get if you pack coins on a table as densely as possible.) Note that the two constellations are scaled so that they occupy the same "area" for the same number of points.



$$d_1 = 1$$

$$d_2 = \frac{\sqrt{2}}{3^{1/4}}$$

$$h_2 = \frac{3^{1/4}}{\sqrt{2}}$$

1. [5 points] Give the exact probability of error for an interior point in constellation $\mathcal{S}_1$.
2. [5 points] What is the **best** upper bound on the probability of error for an interior point of $\mathcal{S}_1$ which you can give using the union bounding technique?
3. [5 points] What is the **best** upper bound on the probability of error for an interior point of $\mathcal{S}_2$ which you can give using the union bounding technique?
4. [5 points] Assume σ^2 is very small. Which constellation would you prefer and why? [Hint: Note that for large $x > 0$, $Q(x)$ behaves essentially like $e^{-\frac{x^2}{2}}$.]