Midterm Exam

Last Name	First Name

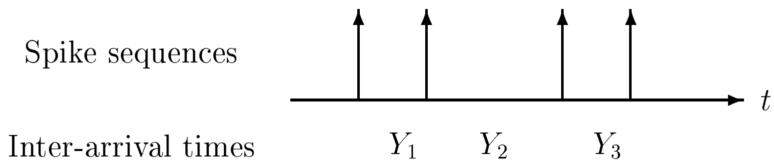
Problem	Points	out of
Problem 1		35
Problem 2		40
Problem 3		25
Total		100

Remarks.

- You have from 14h15 till 15h55 (i.e. 100 minutes) to complete the exam.
- This is a closed-book exam.
- There are 3 problems on the exam.
- The problems are not in order of difficulty.
- A correct answer does not guarantee full credit and a wrong answer does not guarantee loss of credit. You should concisely indicate your reasoning and show all relevant work. The grade on each problem is based on our judgment of your level of understanding as reflected by what you have written.
- Try to do side calculations on a separate sheet and to report on this exam sheet well-organized solutions. If we can't read it, we can't grade it.
- If you don't understand a problem, please ask.

Problem 1 (35 Points) You are asked to develop a "lie detector" and analyze its performance. Based on the observation of brain cell activity, your detector has to decide if a person is telling the truth or is lying.

For the purpose of this problem, the brain cell produces a sequence of spikes as shown in the figure. For your decision you may use only a sequence of n consecutive inter-arrival times $Y_1, Y_2, \ldots, Y_n$. Hence Y_1 is the time elapsed between the first and second spike, Y_2 the time between the second and third, etc.



We assume that, a priori, a person lies with some known probability p. When the person is telling the truth, $Y_1, \ldots, Y_n$ is an i.i.d. sequence of exponentially distributed random variables with intensity α, $(\alpha > 0)$, i.e.

$$f_{Y_i}(y) = \alpha e^{-\alpha y}, \quad y \geq 0.$$

When the person lies, $Y_1, \ldots, Y_n$ is i.i.d. exponentially distributed with intensity β, $(\alpha < \beta)$.

(a) Describe the decision rule of your lie detector for the special case $n = 1$. Your detector shall be designed so as to minimize the probability of error.

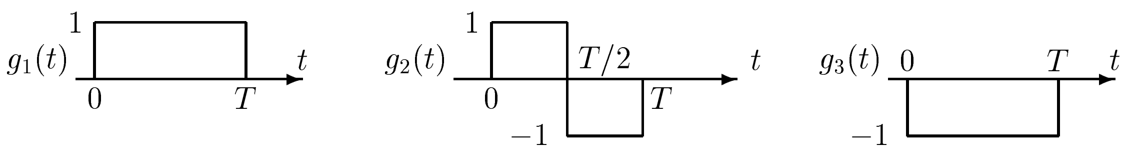
(b) What is the probability $P_{L/T}$ that your lie detector says that the person is lying when the person is telling the truth?

(c) What is the probability $P_{T/L}$ that your test says that the person is telling the truth when the person is lying.

(d) Repeat (a) and (b) for a general n. Hint: There is no need to repeat every step of your previous derivations.

Solution Problem 1

Problem 2 (40 Points) Let $N(t)$ be a zero-mean white Gaussian process of power spectral density $\frac{N_0}{2}$. Let $g_1(t)$, $g_2(t)$, and $g_3(t)$ be waveforms as shown in the following figure.



(a) Determine the norm $\| g_i \|$, $i = 1, 2, 3$.

(b) Let Z_i be the projection of $N(t)$ onto $g_i(t)$. Write down the mathematical expression that describes this projection, i.e. how you obtain Z_i from $N(t)$ and $g_i(t)$.

(c) Describe the object Z_1, i.e. tell us everything you can say about it. Be as concise as you can.

(d) Are Z_1 and Z_2 independent? Justify your answer.

(e) (i) Describe the object $\boldsymbol{Z} = (Z_1, Z_2)$. (We are interested in what it is, not on how it is obtained.)

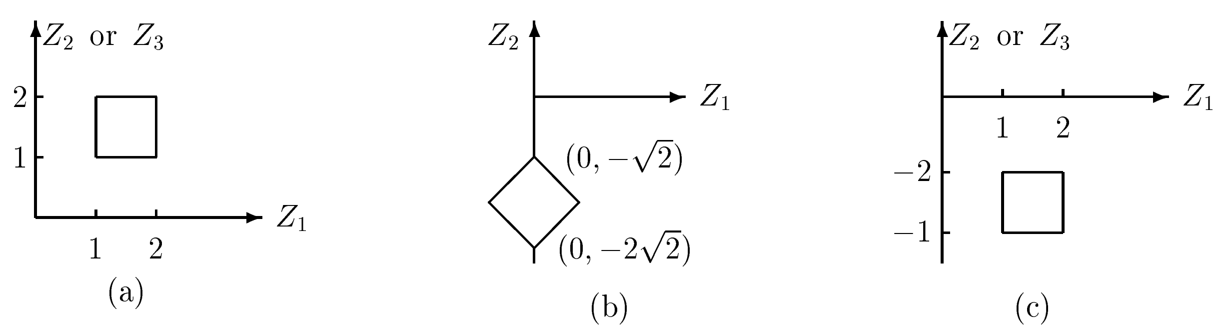
(ii) Find the probability P_a that $\boldsymbol{Z}$ lies in the square labeled (a) in the figure below.

(iii) Find the probability P_b that $\boldsymbol{Z}$ lies in the square (b) of the same figure. Justify your answer.

(f) (i) Describe the object $\boldsymbol{W} = (Z_1, Z_3)$.

(ii) Find the probability Q_a that $\boldsymbol{W}$ lies in the square (a).

(iii) Find the probability Q_c that $\boldsymbol{W}$ lies in the square (c).



Solution Problem 2

Problem 3 (25 Points)

(a) (15 Points) Use Gram Schmidt procedure to find an orthonormal basis for the vector space spanned by the functions shown below. Clearly indicate every step of the procedure. Make sure that $\boldsymbol{s}_1$, $\boldsymbol{s}_2$, and the orthonormal basis are clearly visible.

(b) (10 Points) Let $s(t) = \beta \mathtt{sinc}(\alpha t)$. Plot $s(t)$ (qualitatively but label your plot appropriately) and determine the area $A = \int_{-\infty}^{\infty} s(t)dt$.

Extra Space