MIDTERM EXAM

Closed book

Available Time: 110 minutes

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Print Name

Problem	Score
#1	/ 30
#2	/ 30
#3	/ 40
Total	/ 100

Problem 1 *(Miscellaneous Questions: 30 Points)*

(a) (5 Points) True/False: If X and Y are Gaussian random variables, then (X, Y) is a Gaussian random vector. Explain.

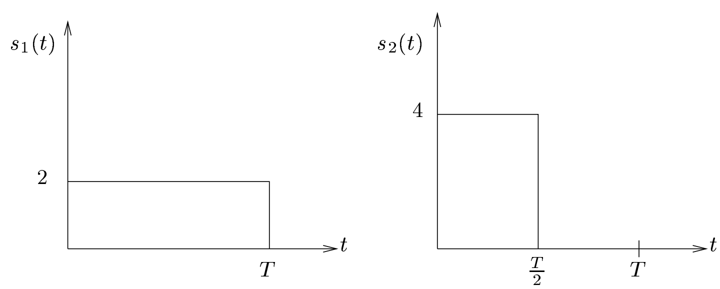
(b) (5 Points) True/False: If X and Y are independent random variables, then they are uncorrelated. Explain.

(c) (5 Points) True/False: If X and Y are uncorrelated random variables, then they are independent. Explain.

(d) (5 Points) True/False: Let $N(t)$ be a white Gaussian noise process of power spectral density $\frac{N_0}{2}$ and let t_0 be an arbitrary time. The random variable $N(t_0)$ has variance $\frac{N_0}{2}$. Explain.

(e) (5 Points) True/False: If $\mathbf{Z} = (Z_1, Z_2) \sim \mathcal{N}(0, I_2)$, the events $Z_1 \in [1, 2]$ and $Z_2 \in [1, 5]$ are independent. Explain.

(f) (5 Points) Draw an orthonormal basis for the vector space spanned by the functions shown below. Also find the corresponding n-tuples $\mathbf{s_1}$ and $\mathbf{s_2}$. Hint: no need to use Gram Schmidt here.



Problem 2 *(ML Receiver and UBB for Orthogonal Signaling: 30 Points)*

Let $H \in \{1, \ldots, m\}$ be uniformly distributed and consider the communication problem described by:

$$H = i: \qquad \mathbf{Y} = \mathbf{s}_i + \mathbf{Z}, \quad \mathbf{Z} \sim \mathcal{N}(0, \sigma^2 I_m),$$

where $\mathbf{s}_1, \ldots, \mathbf{s}_m$, $\mathbf{s}_i \in \mathbb{R}^m$, is a set of constant-energy orthogonal signals. Without loss of generality we assume

$$\mathbf{s}_i = \sqrt{\mathcal{E}}\mathbf{e}_i,$$

where $\mathbf{e}_i$ is the ith unit vector in $\mathbb{R}^m$, i.e., the vector that contains 1 at position i and 0 elsewhere, and $\mathcal{E}$ is some positive constant.

(a) (10 Points) Describe the maximum likelihood decision rule. (Make use of the fact that $\mathbf{s}_i = \sqrt{\mathcal{E}}\mathbf{e}_i$.)

(b) (10 Points) Find the distance $\|\mathbf{s}_i - \mathbf{s}_j\|$.

(c) (10 Points) Upper-bound the error probability $Pr\{e|H = i\}$ using the union bound and the Q function.

Problem 3 *(Minimum Energy for Orthogonal Signaling: 40 Points)*

Let $H \in \{1, \dots, m\}$ be uniformly distributed and consider the communication problem described by:

$$H = i: \qquad \mathbf{Y} = \mathbf{s}_i + \mathbf{Z}, \quad \mathbf{Z} \sim \mathcal{N}(0, \sigma^2 I_m),$$

where $\mathbf{s}_1, \dots, \mathbf{s}_m$, $\mathbf{s}_i \in \mathbb{R}^m$, is a set of constant-energy orthogonal signals. Without loss of generality we assume

$$\mathbf{s}_i = \sqrt{\mathcal{E}}\mathbf{e}_i,$$

where $\mathbf{e}_i$ is the ith unit vector in $\mathbb{R}^m$, i.e., the vector that contains 1 at position i and 0 elsewhere, and $\mathcal{E}$ is some positive constant.

(a) (10 Points) Describe the statistic of Y_j (the jth component of $\mathbf{Y}$) for $j = 1, \dots, m$ given that $H = 1$.

(b) (10 Points) Consider a suboptimal receiver that uses a threshold $t = \alpha\sqrt{\mathcal{E}}$ where $0 < \alpha < 1$. The receiver declares $\hat{H} = i$ if i is the *only* integer such that $Y_i \geq t$. If there is no such i or there is more than one index i for which $Y_i \geq t$, the receiver declares that it can't decide. This will be viewed as an error.

Let $E_i = \{Y_i \geq t\}$, $E_i^c = \{Y_i < t\}$, and describe, in words, the meaning of the event

$$E_1 \cap E_2^c \cap E_3^c \cap \dots \cap E_m^c.$$

(c) (10 Points) Find an upper bound to the probability that the above event *does not* occur when $H = 1$. Express your result using the Q function.

(d) (10 Points) Now we let $\mathcal{E}$ and $\ln m$ go to ∞ while keeping their ratio constant, namely $\mathcal{E} = \mathcal{E}_b \ln m \log_2 e$. (Here $\mathcal{E}_b$ is the energy per transmitted bit.) Find the smallest value of $\mathcal{E}_b/\sigma^2$ (according to your bound) for which the error probability goes to zero as $\mathcal{E}$ goes to ∞. Hint: Use $m - 1 < m = \exp(\ln m)$ and $Q(x) < \frac{1}{2}\exp(-\frac{x^2}{2})$.