ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Principles of Digital Communications: Summer Semester 2007

Assignment date: March 15, 2007
Due date: March 22, 2007

Homework 1

Problem 1. *(Uncorrelation vs. Independence)*

Let X and Y be two random variables.

(i) When are X and Y uncorrelated? When are they independent? Write down the definitions.

(ii) Show that if X and Y are independent, they are also uncorrelated.

(iii) We first define two new random variables U and V:

$$U = \begin{cases} 0 & \text{with prob.} \quad \frac{1}{2} \\ 1 & \text{with prob.} \quad \frac{1}{2}, \end{cases}$$

$$V = \begin{cases} 0 & \text{with prob.} \quad \frac{1}{2} \\ 1 & \text{with prob.} \quad \frac{1}{2}, \end{cases}$$

where U and V are independent. Now, assume that X and Y are defined as follows: $X = U + V$ and $Y = |U - V|$.
Are X and Y independent? Compute the covariance of X and Y. What do you conclude?

Problem 2. *(Properties of the Q Function)*

Prove properties (a) through (d) of the Q function defined in the lecture notes, Section 2.3.

You can use the following hint to help you with property (d):

Hint: Define $\Phi(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$. Then, integrate $\int_x^\infty \Phi(t) \frac{1}{t^2} dt$ by parts.

The resulting equality can be used to prove both inequalities in (d). Start by proving the right-hand inequality.

Problem 3. *(Playing Darts)*

Assume that you are throwing darts at a target. We assume that the target is one-dimensional, *i.e.*, that the darts all end up on a line. The "bulls eye" is in the center of the line, and we give it the coordinate 0. The position of a dart on the target can then be measured with respect to 0.

We assume that the position X_1 of a dart that lands on the target is a random variable that has a Gaussian distribution with variance σ_1^2 and mean 0.

Assume now that there is a second target, which is further away. If you throw dart to that target, the position X_2 has a Gaussian distribution with variance σ_2^2 (where $\sigma_2^2 > \sigma_1^2$) and mean 0.

You play the following game: You toss a coin which gives you "head" with probability p and "tail" with probability $1-p$ for some fixed $p \in [0,1]$. We can model a coin as a Bernoulli random variable Z. If $Z=1$, you throw a dart onto the first target. If $Z=0$, you aim the second target instead. Let X be the relative position of the dart with respect to the center of the target that you have chosen.

(i) Write down X in terms of X_1, X_2 and Z.

(ii) Compute the variance of X.

Bonus question: Is the distribution of X a Gaussian (Note that X is not a linear combination of X_1 and X_2)? Explain.

(iii) Let $S = |X|$ be the score, which is given by the distance of the dart to the center of the target (that you picked using the coin). Compute the average score $\mathbb{E}[S]$.

Problem 4. Let $\boldsymbol{X} \sim \mathcal{N}(0, \sigma^2 I_2)$. For each of the three figures below, express the probability that $\boldsymbol{X}$ lies in the shaded region. You may use the Q-function when appropriate.

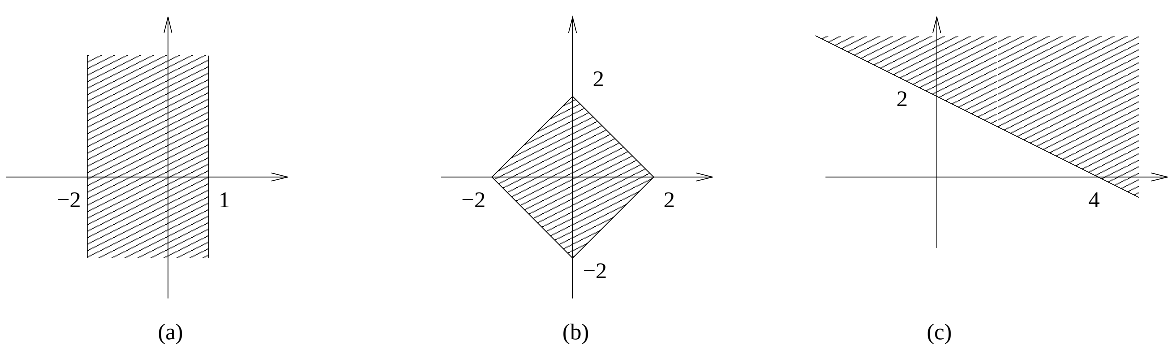

Figure 1: Regions

Problem 5. *(Signal Energy)*

In this problem we give you the opportunity to compute expected values. Even though the exercise may just look to you like a drill, it will turn out to be a useful calculation. In each of the two figures of Figure 2 each point (circle) s_i represents a signal and the square $|s_i|^2$ of the distance between the origin and the point represents the energy of that signal. The axis on the first figure is $\mathbb{R}$, and the axes in the second figure are $\mathbb{R}^2$.

Assume that each signal is used with the same probability.

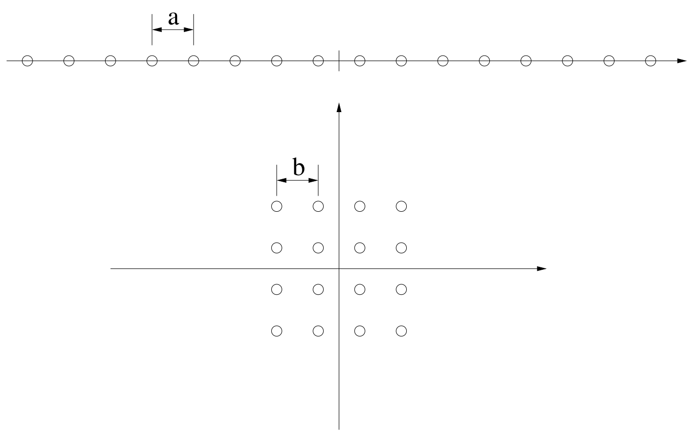

Figure 2: Signals for Problem 5

(i) For each of the two figures compute the average energy $\mathbb{E}[|s_i|^2]$.

(ii) Compute also $\mathbb{E}[s_i]$ for each figure.

Problem 6. *(Simple Hypothesis Testing)*

In this example there are two hypotheses, $H = 0$ and $H = 1$ which occur with probabilities $p_H(0) = p_0$ and $p_H(1) = 1-p_0$, respectively. The observable is $y \in \mathbb{N}_0$, i.e. y is a nonnegative integer. Under hypothesis $H = 0$, y is distributed according to a Poisson law with parameter λ_0, i.e.

$$p_{Y|H}(y|0) = \frac{\lambda_0^y}{y!} e^{-\lambda_0}. \tag{1}$$

Under hypothesis $H = 1$,

$$p_{Y|H}(y|1) = \frac{\lambda_1^y}{y!} e^{-\lambda_1}. \tag{2}$$

This example is in fact modeling the reception of photons in an optical fiber (for more details, see Example 4 of Section 2.2 of class notes).

(i) Derive the MAP decision rule by indicating likelihood and log-likelihood ratios.
Hint: The direction of an inequality changes if both sides are multiplied by a negative number.

(ii) Derive the formula for the probability of error of the MAP decision rule.

(iii) For $p_0 = 1/3$, $\lambda_0 = 2$ and $\lambda_1 = 10$, compute the probability of error of the MAP decision rule. You may want to use a computer program to do this.

(iv) Repeat *(iv)* with $\lambda_1 = 20$ and comment.