Principles of Digital Communication — EPFL, Summer Semester 2006
Prof. Rüdiger Urbanke — July 6, 2006

Final

You have 3 hours. It is not necessarily expected that you finish all problems. Do not loose too much time on each problem but try to collect as many points as possible. **Closed-book, no calculators, cell-phones or crypt sheets. Write only what is relevant to the question!**

Good Luck!!

Name: ______________________________

Prob I	/ 25
Prob II	/ 25
Prob III	/ 25
Prob IV	/ 25
Total	/ 100

Useful Facts:

- $\cos(2x) = 1 - 2\sin^2(x)$

You can write your answers in: English, French, German, Austrian, Italian, Serbian, Dutch, Hindi, Farsi, Tamil, Telugu or Marathi.

Vous trouverez la traduction des problèmes en francais à la fin.

Problem 1[Bandpass Systems/Nyquist Criterion/Sampling Theorem – 25pts]

1. Let the transmitted bandpass signal be of the form

$$x(t) = a\cos(2\pi(f_c + \frac{1}{T})t) + b\cos(2\pi(f_c + \frac{2}{T})t)$$

   where a and b are parameters, $a \in \{0, 1\}$ and $b \in \{0, 1\}$.

   (i) Find the baseband equivalent signal $x_E(t)$ for the transmitted signal.

   (ii) Draw the constellation for the *set* of signals corresponding to all choices of a and b in either baseband or bandpass.

   (iii) If $a = \{0, 1\}$ equally likely and $b = \{0, 1\}$ equally likely find the average energy of the baseband signal. Is this a minimum energy configuration? If not how can you modify the constellation so that it is minimum energy?

2. Mark the correct choice. (correct choice: +2; incorrect choice: -1)

   (ii) Consider the signal $x(t) = \cos(2\pi t)\left(\frac{\sin(\pi t)}{\pi t}\right)^2$. Assume that we sample $x(t)$ with sampling period T. What is the maximum T that guarantees signal recovery?

   a) $T = 1/8$ b) $T = 1/4$ c) $T = 1/2$

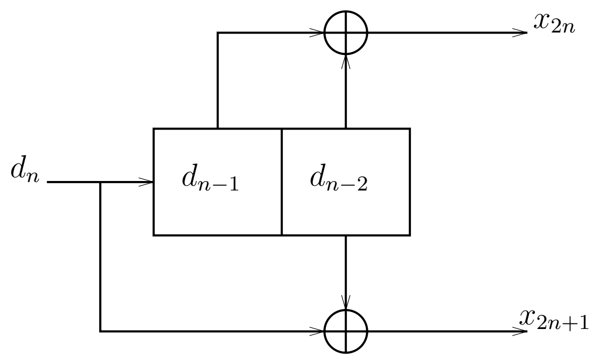
   (iii) Consider the three signals $s_1(t) = 1, s_2(t) = \cos(2\pi t), s_3(t) = \sin^2(\pi t)$, for $0 \le t \le 1$. What is the dimension of the signal space spanned by $\{s_1(t), s_2(t), s_3(t)\}$?

   a) 1 b) 2 c) 3

   (iv) You are given a pulse $p(t)$ with spectrum $p_{\mathcal{F}}(f) = T(1 - |f|T)$, $0 \le |f| \le \frac{1}{T}$. What is the value of $\int p(t)p(t-3T)dt$? (Hint: First think, then calculate!)

   a) 0 b) $3T$ c) $\frac{1}{3T}$

Problem 2[Viterbi/BEC – 25 pts]
Consider the following convolutional encoder. The input sequence belongs to the binary alphabet $\{0, 1\}$. (This means we are using XOR over $\{0, 1\}$ instead of multiplication over $\{\pm 1\}$.)



(i) What is the rate of the encoder?

(ii) Draw one trellis section for the above encoder.

(iii) Consider communication of this sequence through the channel known as Binary Erasure Channel (BEC). The input of the channel belongs to $\{0, 1\}$ and the output belongs to $\{0, 1, ?\}$. The "?" denotes an erasure which means that the output is equally likely to be either 0 or 1. The transition probabilities of the channel are given by

$$P_{Y|X}(0 \mid 0) = P_{Y|X}(1 \mid 1) = 1 - \epsilon,$$
$$P_{Y|X}(? \mid 0) = P_{Y|X}(? \mid 1) = \epsilon.$$

Starting from first principles derive the branch metric of the optimal (MAP) decoder. (Hint: Start with $p(x \mid y)$. Hopefully you are not scared of ∞?)

(iv) Assuming that the initial state is $(0, 0)$, what is the most likely input corresponding to $\{0, ?, ?, 1, 0, 1\}$?

(v) Bonus question: What is the maximum number of erasures the code can correct? (Hint: What is the minimum distance of the code? Just guess from the trellis, don't use the detour graph. :-))

Problem 3[Football – 25pts]
Consider four teams A,B,C,D playing in a football tournament. There are two rounds in the competition. In the first round there are two matches and the winners progress to play in the final. In the first round A plays against one of the other three teams with equal probability $\frac{1}{3}$ and the remaining two teams play against each other. The probability of A winning against any team depends on the number of red cards "r" A gets in the previous match. The probabilities of winning for A against B,C,D denoted by p_b, p_c, p_d are $p_b = \frac{0.5}{(1+r)}, p_c = p_d = \frac{0.6}{1+r}$. In a match against B, A will get 1 red card and in a match against C or D, B will get 2 red cards. Assuming that initially A has 0 red cards and the other teams receive no red cards in the entire tournament and among B,C,D each team has equal chances to win against each other.

Is betting on team A as the winner a good choice ?

Problem 4[Alarm System – 25 pts]
Consider the following model of an home alarm system. The system has a sensor which checks whether a window has been broken. If a window has been broken the sensor ouputs 1 whereas if no window was broken it outputs 0. Let the set of hypotheses be $\mathcal{H} = \{0, 1\}$, where 0 means no break-in and 1 means break-in. Unfortunately, there is some noise Z added to the sensor output. The noise Z has density

$$f_Z(z) = \begin{cases} 1 - |z|, & |z| \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let Y denote the noisy observation. Assume the following priors

$$P_H(0) = \frac{999}{1000}, \qquad P_H(1) = \frac{1}{1000}.$$

Given the sensor output Y, you have to decide whether or not you should call the police to alert them to a possible break-in.

1. Starting from first principles, write down the decision rule which minimizes the probability of error. (Hint: Your decision scheme should take the form of a threshold scheme.)

2. The probability of error is not always the best metric. Note that in this example there are two types of error. You can sound an alarm (call the police) but there is no break-in. Call this the probability of *false-alarm*, p_f. On the other hand you may not sound an alarm but there is a break-in. Call this the *miss probability*, p_m. Assume that in case of a false alarm you have to pay CHF200 to the police for their effort to come to the house. On the other hand, in case you miss a burglary, assume you loose CHF 10000, because of the stolen items.

   (i) Let t be the threshold of your decision scheme. Write down an expression for the *expected* cost that you incur as a function of t. (Hint: Proceed as in question 1. First write down the cost conditioned on $H = 0, 1$.)

   (ii) Minimize the expected cost to find the optimium threshold.