FINAL

You have 3 hours. It is not necessarily expected that you finish all problems. Do not loose too much time on each problem but try to collect as many points as possible.

Closed-book, no calculators, cell-phones or crypt sheets. Write only what is relevant to the question!

Good Luck!!

Name: ______________________________

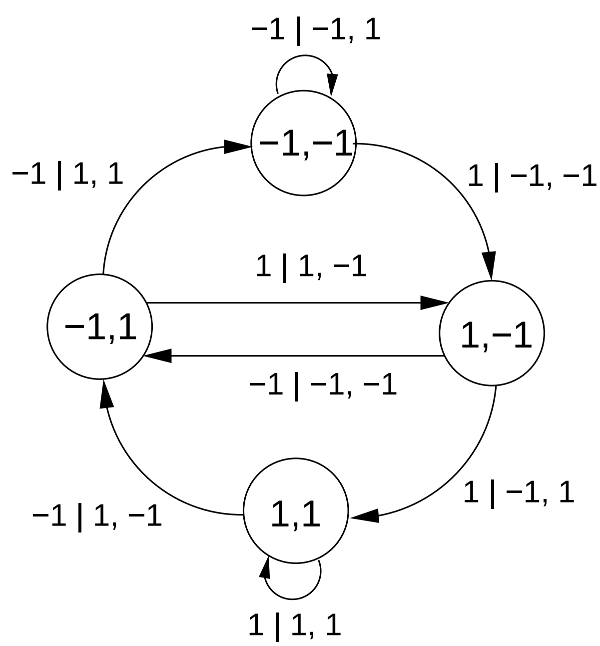
Prob I	/ 25
Prob II	/ 20
Prob III	/ 35
Prob IV	/ 20
Total	/ 100

I. **Convolutional Code and Viterbi Algorithm [25 points]**

Consider a channel, where a transmitter wants to send a sequence $\{D_j\}$ taking values in $\{-1,+1\}$, for $j = 0,1,2,\cdots,k-1$. This sequence is encoded using a convolutional encoder. The channel adds white Gaussian noise to the transmitted signal. If we let X_j denote the transmitted value, then, the received value is: $Y_j = X_j + Z_j$, where $\{Z_j\}$ is sequence of i.i.d. zero-mean Gaussian random variable with variance $\frac{N_0}{2}$. The receiver has to decide which sequence was transmitted using the optimal decoding rule.

a. Convolutional Encoder

Consider the convolutional encoder corresponding to the finite state machine drawn below. The transitions are labeled by $D_j|X_{2j}, X_{2j+1}$, and the states by D_{j-1}, D_{j-2}. We assume that the initial content of the memory is $(1,1)$.



1. What is the rate of this encoder?
3. Sketch the filter (composed of shift registers and multipliers) corresponding to this finite state machine. How many shift registers do you need?
4. Draw a section of the trellis representing this encoder.

b. Viterbi Decoder

Set $k = 3$. Let X_j^i, $j = 0,\cdots,5$, $i = 0,\cdots,7$, denote the output of the convolutional encoder at time j when we transmit hypothesis i. More precisely, we assume that input (information) to the convolutional encoder consists of three bits and that the output (codewords) consists of six bits. No zero bits are appended to the end of the information bits.

Assume that the received vector is $\bar{Y} = (Y_0, Y_1, Y_2, Y_3, Y_4, Y_5) = (-1, -3, -2, 0, 2, 3)$. It is the task of the receiver to decide which hypothesis i was chosen or, equivalently, which vector $\bar{X}^i = (X_0^i, X_1^i, X_2^i, X_3^i, X_4^i, X_5^i)$ was transmitted.

1. Use the Viterbi algorithm to find the most probable transmitted vector $\bar{X}^i$. (Do not loose too much time on this question. If you cannot finish just state the principle.)

c. Performance Analysis.

1. Suppose that this code is decoded using the Viterbi algorithm. Draw the detour flow graph, and label the edges by the input weight using the symbol I, and the output weight using the symbol D.
2. Considering the following generating function

$$T(I, D) = \sum_{i,d} a(i,d) I^i D^d = \frac{ID^4}{1 - 3ID}.$$

Express the quantities

$$\sum_{i,d} i a(i,d) e^{-\frac{d}{2N_0}},$$

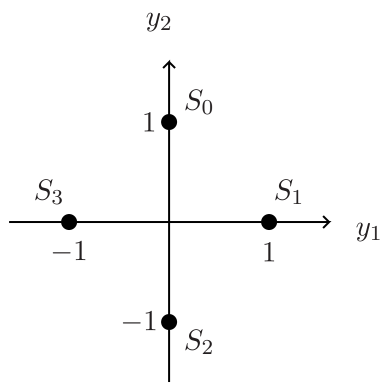
and

$$\sum_{i,d} i(i-1) a(i,d) e^{-\frac{d}{2N_0}},$$

in a compact form in terms of $T(I, D)$. Can you relate any of the quantities to the bit probability of error under Viterbi decoding?

II. **Decision Problem [20 points]**

Consider the following decision problem. The set of hypotheses is $\mathcal{H} = \{0, 1, 2, 3\}$. The prior $p_H(i)$ is uniform. Given $H = i$, $i \in \mathcal{H}$, the transmitter sends the signal S_i. These signals are shown in the figure below. Mathematically, they are given by



$S_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $S_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $S_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, $S_3 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, where all vectors are column vectors. The receiver observes the vector $Y = S_i + Z$, where Z is a zero-mean Gaussian random vector whose covariance matrix is $\Sigma_Z = \begin{pmatrix} 4 & 2 \\ 2 & 5 \end{pmatrix}$. We are interested in the optimum decision rule.

1. Write down the optimum decision rule as a function of Y and Σ_Z.
2. Assume now that instead of observing Y, we observe $\hat{Y} = BY$, where B is a 2-by-2 matrix. This is equivalent to assuming that the channel is described by $\hat{Y} = BS_i + BZ = \hat{S}_i + \hat{Z}$, where we defined $\hat{S}_i = BS_i$ and $\hat{Z} = BZ$. What are the mean and covariance matrix of $\hat{Z}$? What is the distribution of $\hat{Z}$? Write down the optimal decision rule for this new problem as a function of $\hat{Y}$, Σ_Z and B.
3. Let $\Pr_Y\{c\}$ denote the probability of correct decision for the optimal decision rule, assuming that Y is observed, and let $\Pr_{\hat{Y}}\{c\}$ denote the equivalent quantity, assuming that $\hat{Y}$ is observed. For each of the following three matrices B state the relationship between $\Pr_Y\{c\}$ and $\Pr_{\hat{Y}}\{c\}$; justify your response: $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$.
4. Choose a suitable matrix B so that $\Pr_Y\{c\} = \Pr_{\hat{Y}}\{c\}$ and so that the decision problem becomes simple. What are the new transmitted points $\hat{S}_i$ and what is the distribution of $\hat{Z}$? Draw the resulting transmitted points and the decision regions associated to them.

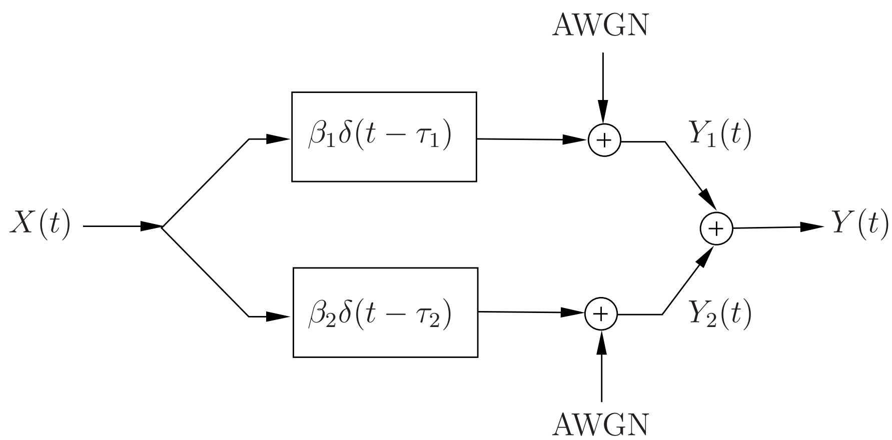
Hint: If $A = \frac{1}{4}\begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix}$, then $A\Sigma_Z A^T = I$, with $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

4. Give a good upper bound to the error probability for this decision problem.

III. **Communication Chain with two receive antennas [35 points]**

Consider the following communications chain, where we have two possible hypotheses H_0 and H_1. The transmitter uses antipodal signaling. To transmit H_0 the transmitter sends a unit energy pulse $p(t)$, and to transmit H_1 he send $-p(t)$. That is, the transmitted signal is $X(t) = \pm p(t)$. The observation consists of $Y_1(t)$ and $Y_2(t)$ as shown below. The signal along each "path" is an attenuated and delayed version of the transmitted signal $X(t)$. The noise is additive white Gaussian with double sided power spectral density $N_0/2$. Also, the noise added to the two observations is independent and independent of the data. The goal of the receiver is to decide which hypotheses was transmitted, based on his observation.

We will look at two different scenarios: either the receiver has access to each individual signal $Y_1(t)$ and $Y_2(t)$, or the receiver has only access to the *combined* observation $Y(t) = Y_1(t) + Y_2(t)$.



a. The case where the receiver has only access to the combined output $Y(t)$.

1. In this case, observe that we can write the received waveform as $\pm g(t) + Z(t)$. What are $g(t)$ and $Z(t)$ and what are the statistical properties of $Z(t)$?

   Hint: Recall that $\int \delta(\tau - \tau_1)p(t-\tau)d\tau = p(t-\tau_1)$.

2. What is the optimal receiver for this case? Your answer can be in the form of a block diagram that shows how to process $Y(t)$ or in the form of equations. In either case, specify how the decision is made between H_0 and H_1.

3. Assume that $\int p(t-\tau_1)p(t-\tau_2)dt = \gamma$, where $-1 \leq \gamma \leq 1$. Find the probability of error for this optimal receiver, expressed it in terms of the Q function, β_1, β_2, γ and $N_0/2$.

b. The case where the receiver has access to the individual antenna outputs $Y_1(t)$ and $Y_2(t)$.

1. Argue that the performance of the optimal receiver for this case can be no worse (*ne peut pas être pire*) than that of the optimal receiver for part (a).
2. Compute the sufficient statistics (Y_1, Y_2), where $Y_1 = \int Y_1(t)p(t - \tau_1)\mathrm{d}t$ and $Y_2 = \int Y_2(t)p(t - \tau_2)\mathrm{d}t$. Show that this sufficient statistic (Y_1, Y_2) has the form $(Y_1, Y_2) = (\beta_1 + Z_1, \beta_2 + Z_2)$ under H_0, and $(Y_1, Y_2) = (-\beta_1 + Z_1, -\beta_2 + Z_2)$ under H_1, where Z_1 and Z_2 are independent zero-mean Gaussian random variables of variance $N_0/2$.
3. Using the LLR (Log-Likelihood Ratio), find the optimum decision rule for this case.

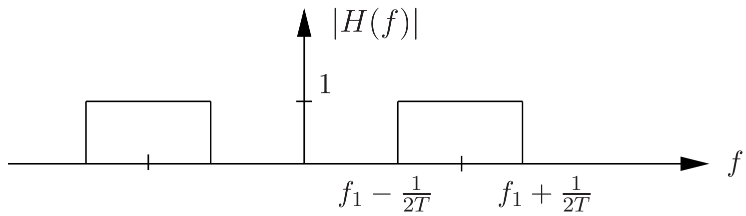
   Hint: It may help to draw the two hypotheses as points in R^2. If we let $V = (V_1, V_2)$ be a Gaussian random vector of mean $m = (m_1, m_2)$ and covariance matrix $\Sigma = \sigma^2 I$, then its pdf is $p_V(v_1, v_2) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{(v_1-m_1)^2}{2\sigma^2} - \frac{(v_2-m_2)^2}{2\sigma^2}\right)$.
4. What is the optimal receiver for this case? Your answer can be in the form of a block diagram that shows how to process $Y(t)$ or in the form of equations. In either case, specify how the decision is made between H_0 and H_1.
5. Find the probability of error for this optimal receiver, expressed in terms of the Q function, β_1, β_2, and N_0.

c. Comparison of the two cases

1. In the case of $\beta_2 = 0$, that is the second observation is solely noise, give the probability of error for both cases (a) and (b). What is the difference between them? Explain why.

IV. **Up-Down Conversion [20 points]**

We want to send a "passband" signal $\psi(t)$ whose spectrum is centered around f_0, through a waveform channel defined by its impulse response $h(t)$. The Fourier transform $H(f)$ of the impulse response is given by



where $f_1 \neq f_0$.

1. Write down, either as a block diagram or in form of equations, the necessary steps to translate the signal $\psi(t)$ into the appropriate frequency range (assuming that the frequency f_0 is translated to frequency f_1).
2. Apply the above frequency translation to the signal
$$\psi(t) = \text{sinc}\left(\frac{t}{T}\right)\cos(2\pi f_0 t).$$
Call the resulting translated signal $\tilde{\psi}(t)$. What is the output signal, assuming that $\tilde{\psi}(t)$ is sent through the given passband channel with impulse response $h(t)$. Specify this output both in passband as well as in baseband.
3. Assume that $f_0 = f_1 + \epsilon$, with $0 \le \epsilon \ll \frac{1}{2T}$. Assume that the signal $\psi(t)$ is sent directly through the passband channel with impulse response $h(t)$, without frequency translation. Determine the center frequency of the resulting signal.

Hint: The Fourier transform of $\cos(2\pi f_0 t)$ is $\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$. The Fourier transform of $\frac{1}{T}\text{sinc}(\frac{t}{T})$ is equal to $\mathbf{1}_{[-\frac{1}{2T}, \frac{1}{2T}]}(f)$ with $\mathbf{1}_{[-\frac{1}{2T}, \frac{1}{2T}]}(f) = 1$ if $f \in [-\frac{1}{2T}, \frac{1}{2T}]$ and 0 otherwise.